A BRANCHING RULE FOR PARTITION COMPLEXES

G. Z. ARONE

ABSTRACT. Let Σ_n be the symmetric group, and let $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \subset \Sigma_n$ be a Young subgroup. Let Π_n be the complex of partitions of $\mathbf{n}$. Our main result is a $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-equivariant decomposition of $|\Pi_n|$. As an application, we obtain new information about the quotient space of $|\Pi_n|$ by a Young subgroup.

1. INTRODUCTION

Let Π_n denote the poset of (proper, nontrivial) partitions of the set $\mathbf{n} = \{1, \ldots, n\}$. Let $|\Pi_n|$ be the geometric realization of Π_n. It is well-known that for $n \geq 3$ there is a homotopy equivalence

$$|\Pi_n| \simeq \bigvee_{(n-1)!} S^{n-3}.$$

However the poset Π_n has a natural action of the symmetric group Σ_n, which is not preserved by this equivalence. The space $|\Pi_n|$, with its natural action of Σ_n, arises in various contexts. For example, the homology group $\mathrm{H}_{n-3}(|\Pi_n|)$ is closely related to the Lie representation.

Let $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \subset \Sigma_n$ be a Young subgroup. In this paper we investigate the $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-equivariant homotopy type of $|\Pi_n|$. To state our result, we need to recall one more definition. Let $\mathcal{L}[x_1, \ldots, x_k]$ be the free Lie algebra (over $\mathbb{Z}$) on k generators. Let $M(n_1, \ldots, n_k) \subset \mathcal{L}[x_1, \ldots, x_k]$ be the subgroup generated by commutators that contain x_i exactly n_i times, for all $1 \leq i \leq k$. It is well-known that $M(n_1, \ldots, n_k)$ is a free Abelian group. The rank of $M(n_1, \ldots, n_k)$ is given by a famous formula due to Witt [26]:

$$(1.1) \qquad \frac{1}{n} \sum_{d \mid \gcd(n_1, \ldots, n_k)} \mu(d) \binom{\frac{n}{d}}{\frac{n_1}{d}, \ldots, \frac{n_k}{d}}.$$

Here d ranges over positive common divisors of $n_1, \ldots, n_k$. Later we will use a particular basis $B(n_1, \ldots, n_k)$ of $M(n_1, \ldots, n_k)$. For now, all that matters is that $B(n_1, \ldots, n_k)$ is a finite set with size given by Witt's formula (1.1).

The following is our main theorem. By a G-equivariant equivalence we mean a map of G spaces that induces an equivalence of H-fixed points for every subgroup H of G.

Theorem 1.2. *Suppose $n = n_1 + \cdots + n_k$. Let $g = \gcd(n_1, \ldots, n_k)$. There is a $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-equivariant homotopy equivalence*
(1.3)

$$|\Pi_n| \longrightarrow \bigvee_{d|g} \left(\bigvee_{B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})} \Sigma_{n_1} \times \cdots \times \Sigma_{n_k +} \wedge_{\Sigma_d} S^{n-d-1} \wedge |\Pi_d|^\diamond \right).$$

Here d ranges over positive divisors of g.

Here is a brief explanation of the terms in equation (1.3). Throughout the paper, $n = n_1 + \cdots + n_k$, and $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \subset \Sigma_n$ is the subgroup of permutations that leave invariant the blocks

$$\{1, \ldots, n_1\}, \{n_1+1, n_1+2, \ldots, n_1+n_2\}, \ldots, \{n_1+\cdots+n_{k-1}+1, \ldots, n\}.$$

If $d > 0$ divides all of $n_1, \ldots, n_k$, then d divides n, and

$$\underbrace{\Sigma_d \times \cdots \times \Sigma_d}_{\frac{n}{d}}$$

is a subgroup of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$. We have group inclusions

$$\Sigma_d \hookrightarrow \Sigma_d^{\frac{n}{d}} \hookrightarrow \Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \hookrightarrow \Sigma_n.$$

We identify Σ_d with its image under each one of these inclusions.

If X is a topological space, $X^\diamond$ denotes the unreduced suspension of X. We have to use the unreduced suspension of $|\Pi_d|$ in the main theorem, because $|\Pi_d|$ does not have a Σ_d-invariant basepoint. However, note that if $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$ is a proper Young subgroup of Σ_n, then $|\Pi_n|$ does have a $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-invariant basepoint. The map (1.3) is in fact a pointed equivariant homotopy equivalence.

By S^{n-d-1} we mean a desuspension of the sphere $S^{n-d} = (S^d)^{\wedge \frac{n-d}{d}}$, equipped with the natural action of Σ_d. Note that since d divides n, it divides $n - d$, so this is well-defined. There is a Σ_d-equivariant homeomorphism $S^d \cong S^1 \wedge \widehat{S}^{d-1}$, where on the right hand side Σ_d acts trivially on S^1, and acts on $\widehat{S}^{d-1}$ via the reduced standard representation. It follows that there is a Σ_d-equivariant homeomorphism

$$S^{n-d-1} \cong S^{\frac{n-d}{d}-1} \wedge (\widehat{S}^{d-1})^{\wedge \frac{n-d}{d}}.$$

It follows that $S^{n-d-1} \wedge |\Pi_d|^\diamond$ is a space with a pointed action of Σ_d. The space

$$\Sigma_{n_1} \times \cdots \times \Sigma_{n_k +} \wedge_{\Sigma_d} S^{n-d-1} \wedge |\Pi_d|^\diamond$$

is homeomorphic to a wedge sum of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}/\Sigma_d$ copies of $S^{n-d-1} \wedge |\Pi_d|^\diamond$, and it has a natural action of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$.

We should note that our definition of Π_d only works well for $d \geq 2$. For $d = 2$, $|\Pi_2|$ is the empty set, which we identify with the -1-dimensional sphere, since its unreduced suspension $|\Pi_2|^\diamond$ is homeomorphic to S^0. We decree that $|\Pi_1|$ is the -2-dimensional sphere. In practice this means that the space $|\Pi_1|$ is undefined, but for $l \geq 0$, $S^l \wedge |\Pi_1|^\diamond := S^{l-1}$. In particular, the wedge summand corresponding to $d = 1$ in equation (1.3) is equivalent to

$$\bigvee_{B(n_1,\ldots,n_k)} \Sigma_{n_1} \times \cdots \times \Sigma_{n_k+} \wedge S^{n-3}.$$

Examples 1.4. If $\gcd(n_1, \ldots, n_k) = 1$, Theorem 1.2 says that $|\Pi_n|$ is $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-equivariantly equivalent to a wedge sum of copies of S^{n-3}, that are freely permuted by $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$. In the special case of the subgroup $\Sigma_{n-1} \times \Sigma_1 \subset \Sigma_n$, we note that $|B(n-1,1)| = 1$ and conclude that that there is a Σ_{n-1}-equivariant homotopy equivalence

$$|\Pi_n| \simeq_{\Sigma_{n-1}} \Sigma_{n-1+} \wedge S^{n-3}.$$

This equivalence is a well-known result. For example, see [8] for a closely related statement.

In the case of the subgroup $\Sigma_{n-2} \times \Sigma_2$, where n is even, one easily calculates that $|B(n-2,2)| = \frac{n-2}{2}$ and $|B(\frac{n-2}{2},1)| = 1$. We conclude that there is a $\Sigma_{n-2} \times \Sigma_2$-equivariant equivalence

$$(1.5) \quad |\Pi_n| \simeq \left(\bigvee_{\frac{n-2}{2}} \Sigma_{n-2} \times \Sigma_{2+} \wedge S^{n-3} \right) \vee \Sigma_{n-2} \times \Sigma_{2+} \wedge_{\Sigma_2} S^{n-3}.$$

Note that in the last summand S^{n-3} has an action of Σ_2: it is homeomorphic to the smash product $S^{\frac{n}{2}-2} \wedge (\hat{S}^1)^{\wedge \frac{n}{2}-1}$, where $S^{\frac{n}{2}-2}$ is a sphere with a trivial action of Σ_2 while $(\hat{S}^1)^{\wedge \frac{n}{2}-1}$ is a sphere on which Σ_2 acts by the suspension of the antipodal action. Formula (1.5) was first obtained by Ragnar Freij by different methods. See [11], Theorems 5.3 and 5.5.

In the case $n = 4$ we obtain a $\Sigma_2 \times \Sigma_2$-equivariant equivalence

$$|\Pi_4| \to \Sigma_2 \times \Sigma_{2+} \wedge S^1 \vee \Sigma_2 \times \Sigma_{2+} \wedge_{\Sigma_2} \hat{S}^1$$

where $\hat{S}^1$ denotes a circle on which Σ_2 acts by reflection. This equivalence is illustrated in Figure 1.4. $|\Pi_4|$ is a one-dimensional complex. The map is defined by collapsing to a point a $\Sigma_2 \times \Sigma_2$-equivariantly contractible subcomplex, depicted with thin lines. The subcomplex is drawn separately on the left in a form that makes its contractibility

FIGURE 1. The $\Sigma_2 \times \Sigma_2$-equivariant equivalence $|\Pi_4| \to$
$\Sigma_2 \times \Sigma_{2+} \wedge S^1 \vee \Sigma_2 \times \Sigma_{2+} \wedge_{\Sigma_2} \hat{S}^1$ is defined by collapsing
the contractible subcomplex drawn with thin lines. The
collapsed subcomplex is drawn separately on the left.
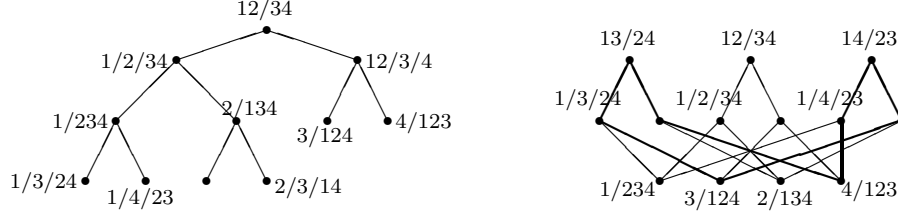
apparent. The reader is invited to check that the subcomplex of thin
lines is $\Sigma_2 \times \Sigma_2$-invariant, and that the quotient space of $|\Pi_4|$ by this
subcomplex is homeomorphic to $\Sigma_2 \times \Sigma_{2+} \wedge S^1 \vee \Sigma_2 \times \Sigma_{2+} \wedge_{\Sigma_2} \hat{S}^1$.

Our Theorem 1.2 is a strengthening of the main result of [4]. There
is an overlap of methods as well. In both papers, the proof uses the
relationship between $|\Pi_n|$ and Lie algebras, along with a decomposition
result for free groups, or free Lie algebras. In [loc. cit], the authors
used the Hilton-Milnor theorem, which is a decomposition result for the
topological free group generated by a wedge sum of connected spaces.
Using the Hilton-Milnor theorem, it was possible to prove a weak ver-
sion of Theorem 1.2. In this paper we use an algebraic version of the
Hilton-Milnor theorem, which is a decomposition result for a free Lie
algebra on a direct sum. Using this algebraic version enables us to have
a point-set level description of the map in (1.3), and to prove that it is
an equivariant homotopy equivalence.

The proof of the main theorem also depends on describing the fixed
points space $|\Pi_n|^G$ for certain subgroups $G \subset \Sigma_n$. Then once proved,
Theorem 1.2 can be used to describe the fixed points space for *all*
subgroups $G \subset \Sigma_n$, in terms of posets of subgroups. These results are
not really new (see Remark 1.9 below), but they are perhaps not widely
known, so we will give a brief outline.

Suppose G acts effectively on $\mathbf{n}$. Each orbit of the action of G on
$\mathbf{n}$ is a G-invariant subset of $\mathbf{n}$. We say that the action of G on $\mathbf{n}$ is
isotypical if all the orbits of G are pairwise isomorphic as G-sets. The
following is proved in the paper as Lemma 7.1.

Lemma 1.6. *Suppose G acts on $\mathbf{n}$, and the action is not isotypical.
Then $|\Pi_n|^G \simeq *$.*

We learned recently that this lemma was observed independently by Markus Hausmann. We suspect that it had been observed by others as well, but we do not know of a published reference.

It remains to analyze the fixed points of isotypical actions. An important special case of isotypical actions is that of transitive actions. If G acts transitively on $\mathbf{n}$, then $\mathbf{n}$ can be identified, as a G-set, with the coset space G/H, where H is a subgroup of G (determined up to conjugacy). For transitive actions, it is not difficult to show that the poset of G-invariant partitions of G/H is isomorphic to the poset of subgroups of G that contain H. This is Lemma 7.2 (it had appeared in the paper of White-Williamson [25, Lemma 3], and they attributed the result to Klass [15]).

Using the transitive case together with the main theorem, one can describe the fixed point space $|\Pi_n|^G$ for a general isotypical action of G on $\mathbf{n}$ as follows. Suppose G acts isotypically on $\mathbf{n}$. Then there exists a unique d dividing n such that G is a transitive subgroup of Σ_d, embedded diagonally in Σ_n. Our description of $|\Pi_n|^G$ depends on this d. The following is proved in the paper as 9.1.

Proposition 1.7. *Suppose that Σ_d is embedded diagonally in Σ_n and G is a transitive subgroup of Σ_d. Let $C_{\Sigma_d}(G)$ be the centralizer of G in Σ_d. There is an equivalence ($*$ denotes join)*

$$|\Pi_n|^G \simeq C_{\Sigma_d}(G)_+^{\frac{n}{d}} \wedge_{C_{\Sigma_d}(G)} \Sigma |\Pi_d|^G * |\Pi_{\frac{n}{d}}|.$$

Example 1.8. There is some overlap between this part of the paper and [3]. In that paper, the authors analyzed the fixed point space of $|\Pi_n|^P$ where P is a p-subgroup of Σ_n. More precisely, it was shown that the only p-groups P for which the fixed points space $|\Pi_n|^P$ is not contractible are elementary abelian groups acting freely on $\mathbf{n}$. It also was observed in [loc. cit.] that if $n = p^k$ and $P \cong (\mathbb{Z}/p)^k$ acts freely and transitively on $\mathbf{n}$, then $|\Pi_n|^P$ is isomorphic to the poset of proper non-trivial subgroups of P, which is closely related to the Tits building for $\mathrm{GL}_k(\mathbb{F}_p)$. We complete the calculation in Corollary 9.3, where we describe the fixed points for elementary abelian groups that act freely but not necessarily transitively.

Remark 1.9. After making this paper public we learned that our results about fixed points of the partition poset (in particular Proposition 1.7 and its consequences) were discovered independently by Lukas Brantner in the spring of 2014. He proved Proposition 1.7 directly, using discrete Morse theory [7].

In addition to fixed points, we consider quotient spaces of $|\Pi_n|$ by an action of a Young subgroup. Here we seem to have something genuinely new to say. Quotient spaces of $|\Pi_n|$ by subgroups of Σ_n have received some attention over the years. For starters, it is known that the quotient space of $|\Pi_n|$ by the full symmetric group is contractible for $n \geq 3$ (see Kozlov [16]). Results about quotient spaces by certain subgroups of $|\Sigma_n|$ were obtained, for example, by Patricia Hersh [14] and Ralf Donau [8]. Here we observe that Theorem 1.2 has the following immediate consequence regarding quotient by action of a Young subgroup.

Proposition 1.10. *Suppose* $n = n_1 + \cdots + n_k$. *Let* $g = \gcd(n_1, \ldots, n_k)$. *There is a homotopy equivalence*

$$\Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \backslash |\Pi_n| \longrightarrow \bigvee_{d|g} \bigvee_{B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})} \Sigma_d \backslash \left(S^{n-d-1} \wedge |\Pi_d|^\diamond \right).$$

Proposition 1.10 reduces the problem of calculating quotient spaces of Π_n by Young subgroups of Σ_n to the problem of calculating quotient spaces of the form

$$(1.11) \qquad\qquad \Sigma_d \backslash S^{ld-1} \wedge |\Pi_d|^\diamond.$$

Here $l \geq 1$, and S^{ld-1} is equipped with an action of Σ_d: desuspension of the natural action on S^{ld}. The following proposition summarizes our results about these quotient spaces.

Proposition 1.12. *The quotient space* $\Sigma_d \backslash S^{ld-1} \wedge |\Pi_d|^\diamond$ *is:*
  (1) *rationally contractible unless* $d = 1$ *or* $d = 2$ *and* l *is even.*
  (2) *equivalent to* $\Sigma^l \mathbb{R}P^{l-1}$ *if* $d = 2$.
  (3) *equivalent to the p-localization of the orbit space* $\Sigma_p \backslash S^{lp-1}$ *if* $d = p = $ *an odd prime, and* l *is odd.*
  (4) *equivalent to the p-localization of the homotopy cofiber of the quotient map* $S^{lp-1} \to \Sigma_p \backslash S^{lp-1}$ *if* $d = p = $ *an odd prime, and* l *is even.*
  (5) *contractible if* $l = 1$
  (6) *contractible if* $l = 2$ *and* d *is an odd prime.*

In other words, we manage to analyze $\Sigma_d \backslash S^{ld-1} \wedge |\Pi_d|^\diamond$ when d is a prime (or $l = 1$). We have no doubt that more complicated values of d are within reach, and we are hopeful that eventually these spaces will be well-understood for all l and d. In the meantime, propositions 1.10 and 1.12 enable us to
  (1) describe the torsion-free part of $H_*(\Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \backslash |\Pi_n|)$ in all cases (but this is not new). This is Corollary 10.8.

(2) prove that $\widetilde{\mathrm{H}}_*({}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}\backslash|\Pi_n|)$ has p-primary torsion only for primes p that satisfy $p \le \gcd(n_1, \ldots, n_k)$ (Lemma 10.9).
(3) describe the homotopy type of ${}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}\backslash|\Pi_n|$ when $\gcd(n_1, \ldots, n_k)$ equals 1 or a prime number (Corollary 10.2 and Proposition 10.15).
(4) describe the homotopy type of ${}_{\Sigma_{p^2} \times \Sigma_{p^2}}\backslash|\Pi_{2p^2}|$ where p is a prime.
(5) classify all Young subgroups for which the quotient space is equivalent to a wedge of spheres (Corollary 10.18)

Organization of the paper. In Section 2 we review background material on simplicial complexes (with emphasis on the partition complex), the join construction and equivariant topology. In Section 3 we review some generalities regarding free Lie algebras, and state the algebraic Hilton-Milnor theorem.

In Section 4 we show how the algebraic Hilton-Milnor theorem gives rise to a branching rule for the Lie representation. This is an algebraic analogue of our main theorem.

In Section 5 we review Barcelo's theorem on the isomorphism between the Lie representation and the cohomology of the partition complex. In Section 6 we define the map (1.3), and prove that it induces an isomorphism in cohomology, by comparing it with the isomorphism that we constructed earlier in Section 4.

In Section 7 we analyze the fixed point space of $|\Pi_n|$ for some groups, including all groups that occur as an isotropy of either the source or target of the map (1.3). In Section 8 we use the calculations in the preceding section to show that the map in (1.3) induces an equivalence of fixed points spaces for every group that occurs as an isotropy of either the source or target of the map. We conclude that the map is an equivariant equivalence, thus proving the main theorem.

In Section 9 we use the main theorem to describe (in terms of subgroup posets) the fixed point space $|\Pi_n|^G$ for every $G \subset \Sigma_n$. In Section 10 we use the main theorem to study the quotient space of $|\Pi_n|$ by an action of a Young subgroup.

2. Preliminaries

2.1. **Simplicial complexes and posets.** We will assume that the reader is familiar with basic notions concerning simplicial complexes, the order complex of a poset, and geometric realizaiton. Given a poset $\mathcal{P}$, let $|\mathcal{P}|$ denote its geometric realization.

Many posets have an initial and/or final element. We will denote the initial and final element of a poset $\mathcal{P}$ by $\hat{0}$ and $\hat{1}$, or $\hat{0}_{\mathcal{P}}$ and $\hat{1}_{\mathcal{P}}$ if needed. Given a poset $\mathcal{P}$, we denote the poset $\mathcal{P} \setminus \{\hat{0}, \hat{1}\}$ by $\overline{\mathcal{P}}$.

Example 2.1. Let $\mathcal{C}_m$ be the poset of integers between 1 and m, with the usual linear order. Then $|\mathcal{C}_m| = \Delta^{m-1}$ is the standard $m-1$-dimensional simplex. The simplicial boundary of $\mathcal{C}_m$ is the simplicial complex consisting of all the increasing chains in $\mathcal{C}_m$, except for the unique maximal chain $1 < \cdots < m$. Its geometric realization is the topological boundary of Δ^{m-1}, i.e. the sphere S^{m-2}.

Example 2.2. Let $\mathcal{B}_m$ be the poset of subsets of $\mathbf{m}$. It is easy to see that there is an isomorphism of posets $\mathcal{B}_m \cong (\mathcal{B}_1)^m$. Since $|\mathcal{B}_1| \cong [0,1]$, there is a homeomorphism $|\mathcal{B}_m| \cong [0,1]^m$. Next, consider the posets $\mathcal{B}_m \setminus \{\hat{0}\}$ – the poset of non-empty subsets of $\mathbf{m}$, and $\overline{\mathcal{B}_m}$ – the poset of proper, non-empty subsets of $\mathbf{m}$. It is well-known, and not difficult to check, that the order complexes of $\mathcal{B} \setminus \{\hat{0}\}$ and $\overline{\mathcal{B}_m}$ are isomorphic to the barycentric subdivisions of $\mathcal{C}_m$ and of its simplicial boundary, respectively (see Example 2.1). In particular, there is a homeomorphism $|\overline{\mathcal{B}_m}| \cong S^{m-2}$. The action of Σ_m on $\mathcal{B}_m$ induces an unpointed action on $|\overline{\mathcal{B}_m}|$, i.e. on S^{m-2}. Once again, it is well-known and not difficult to prove that this action is homeomorphic to the action of Σ_m on the unit sphere in $\mathbb{R}^{m-1}$, where $\mathbb{R}^{m-1}$ is endowed with the reduced standard action of Σ_m.

Given two elements x and y of a poset $\mathcal{P}$, their least upper bound, if it exists, is denoted $x \vee y$. The greatest lower bound is denoted $x \wedge y$. In the course of the paper we will use several times the following elementary criterion for the contractibility of a poset.

Lemma 2.3. *Suppose $\mathcal{P}$ has an element θ such that $x \vee \theta$ exists for all $x \in \mathcal{P}$ (or alternatively, $x \wedge \theta$ exists for all x). Then $|\mathcal{P}|$ is contractible.*

Proof. The assumption implies that there is a well-defined map of posets $- \vee \theta \colon \mathcal{P} \to \mathcal{P}$. This map has the property that $x \leq x \vee \theta$ for all $x \in \mathcal{P}$. In other words, there is a natural transformation from the identity functor to $- \vee \theta$. It is well-known that a natural transformation between functors induces a homotopy between maps of geometric realizations. So the induced map of geometric realizations $|- \vee \theta| \colon |\mathcal{P}| \to |\mathcal{P}|$ is homotopic to the identity map. But this map

factors through $|\mathcal{P}_{\geq\theta}|$, where $\mathcal{P}_{\geq\theta} \subset \mathcal{P}$ is the subposet of all elements x satisfying $x \geq \theta$. This poset has an initial element, namely θ, so $|\mathcal{P}_{\geq\theta}|$ is contractible. It follows that the map $|-\vee\theta|$ is null-homotopic, and therefore the identity map of $|\mathcal{P}|$ is null-homotopic. It follows that $|\mathcal{P}|$ is contractible. The proof under the assumption of existence $x \wedge \theta$ is similar. $\qquad\square$

2.1.1. *The join operation on spaces and posets.* Next we will review the join operation, since it plays a role in this paper. First recall that the join of two topological spaces X and Y, denoted $X * Y$, is the double mapping cylinder (a.k.a homotopy pushout) of the diagram

$$X \leftarrow X \times Y \rightarrow Y.$$

It is easy to check that the operation is associative up to natural isomorphism. It is well-known that if X and Y are pointed spaces, there is a natural homotopy equivalence

$$(2.4) \qquad\qquad X * Y \xrightarrow{\simeq} S^1 \wedge X \wedge Y.$$

This fact can be generalized slightly to a situation where only one of the spaces X and Y has a basepoint. Recall that $X^\diamond$ denotes the unreduced suspension of X. This space has a canonical basepoint (the "south pole"). We note that $X^\diamond$ is homeomorphic to the pointed homotopy cofiber of the canonical map $X_+ \rightarrow S^0$.

Lemma 2.5. *Suppose Y is a well-pointed space. Then there is a natural weak equivalence*

$$X * Y \xrightarrow{\simeq} X^\diamond \wedge Y.$$

Proof. Consider the following diagram

$$
\begin{array}{ccccc}
X & \leftarrow & X \times * & \rightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
X & \leftarrow & X \times Y & \rightarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
* & \leftarrow & X_+ \wedge Y & \rightarrow & Y
\end{array}
$$

The columns in the diagram are homotopy cofibration sequences, so it induces a homotopy cofibration sequence of homotopy pushouts of the rows. The homotopy pushout of the top row is contractible, so we have an equivalence between the pushouts of the middle and bottom rows. The pushout of the middle row is $X * Y$. The bottom row is homeomorphic to the diagram obtained by taking object-wise smash product of the following diagram with Y

$$* \leftarrow X_+ \rightarrow S^0.$$

Therefore, the homotopy pushout of the bottom row is the smash product of the unpointed homotopy cofiber of the map $X_+ \to S^0$ with Y. Since the spaces X_+, S^0 and Y are well-pointed, the unpointed homotopy cofiber can be replaced with the pointed homotopy cofiber. $\square$

It follows that the expression $S^{n-d-1} \wedge |\Pi_d|^\diamond$ in the main theorem (see (1.3)) can be replaced with $S^{n-d-1} * |\Pi_d|$. This form seems to be more convenient for the purpose of proving the theorem.

Examples 2.6. The empty set is a unit for the join operation. That is, there are natural isomorphisms $X * \emptyset \cong \emptyset * X \cong X$.

Taking join with S^0 is the same as taking unreduced suspension. We use notations $S^0 * X$, $X^\diamond$ and ΣX interchangeably.

A special case of Lemma 2.5 is the well-known equivalence $S^m * S^n \cong S^{m+n+1}$ (in fact these spaces are homeomorphic). We think of the empty set as the -1-dimensional sphere. Then the formula remains valid for $m, n \geq -1$.

The join of two simplices $\Delta^i * \Delta^j$ is homeomorphic to Δ^{i+j+1}. It also is easy to see that $\Delta^0 * S^j$, the join of a point and a sphere, is homeomorphic to Δ^{j+1}. It follows that for all $i \geq 0$, $j \geq -1$, $\Delta^i * S^j$ is homeomorphic to Δ^{i+j+1}.

There is a version of the join operation for simplicial complexes. Let (U, Φ) and (V, Ψ) be simplicial complexes. Their simplicial join is defined to be the simplicial complex $(U, \Phi) * (V, \Psi) := (U \coprod V, \Phi * \Psi)$, where $\Phi * \Psi$ is the collection of subsets $S \subset U \coprod V$ satisfying $S \cap U \in \Phi$ and $S \cap V \in \Psi$. It is well known that geometric realization takes simplicial join to space-level join.

Now let $\mathcal{P}$ and $\mathcal{Q}$ be posets. The join of $\mathcal{P}$ and $\mathcal{Q}$ is defined to be the following poset. On the level of underlying sets, $\mathcal{P} * \mathcal{Q} = \mathcal{P} \coprod \mathcal{Q}$. The partial order is defined by keeping the old order on $\mathcal{P}$ and $\mathcal{Q}$, and decreeing that every element of $\mathcal{P}$ is smaller than every element of $\mathcal{Q}$. It is easy to see that the order complex of $\mathcal{P} * \mathcal{Q}$ is isomorphic to the simplicial join of the two order complexes. It follows that $|\mathcal{P} * \mathcal{Q}| \cong |\mathcal{P}| * |\mathcal{Q}|$.

There is another model for the join of two posets, that will be useful to us. First, some notation. For an element $a \in \mathcal{P}$, let $\mathcal{P}_{<a} = \{x \in \mathcal{P} \mid x < a\}$. Define $\mathcal{P}_{>a}, \mathcal{P}_{\leq a}$, and $\mathcal{P}_{\geq a}$ analogously. Suppose $b \in \mathcal{Q}$. Then $(a, b) \in \mathcal{P} \times \mathcal{Q}$. The following lemma is due to Quillen [21, Proposition 1.9]. See also [24, Theorem 5.1 (b), (c)].

Lemma 2.7. *There are natural homeomorphisms*

$$|(\mathcal{P} \times \mathcal{Q})_{>(a,b)}| \cong |\mathcal{P}_{>a}| * |\mathcal{Q}_{>b}|$$

$$|(\mathcal{P} \times \mathcal{Q})_{<(a,b)}| \cong |\mathcal{P}_{<a}| * |\mathcal{Q}_{<b}|$$

Let $\mathcal{P}$ be a poset with an initial and final element. Recall that $\overline{\mathcal{P}} = \mathcal{P} \setminus \{\hat{0}, \hat{1}\}$. Suppose that $\overline{\mathcal{P}} = \emptyset$. Then there are two possibilities. Either $\mathcal{P}$ has two elements, or just one. In the first case, we identify $|\overline{\mathcal{P}}|$ with the empty space, as expected. But if $\mathcal{P}$ has just one element, which is then both initial and final, then we adopt the convention that $|\overline{\mathcal{P}}| = S^{-2}$ (recall that we identify the empty space with S^{-1}). In practice this means that the space $|\overline{\mathcal{P}}|$ is not defined, but any suspension of this space is. In particular $S^0 * |\overline{\mathcal{P}}| = S^{-1} = \emptyset$, and in general $S^n * |\overline{\mathcal{P}}| \cong S^{n-1}$.

With this convention in place, we have the following formula for the "product of closed intervals, with initial and final object removed".

Proposition 2.8. *Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be posets, each with an initial and final object. There is a natural homeomorphism*

$$|\overline{\mathcal{P}_1 \times \cdots \times \mathcal{P}_n}| \cong S^{n-2} * |\overline{\mathcal{P}_1}| * \cdots * |\overline{\mathcal{P}_n}|.$$

Moreover, a permutation of factors on the left hand side corresponds on the right hand side to permutation of factors, and the standard action of a permutation on S^{n-2} (as in Example 2.2).

Proof. The case $n = 2$ is [24, Theorem 5.1 (d)]. The general case is proved by a similar argument, plus induction. $\square$

Once again, note that if for some i the poset $\mathcal{P}_i$ consists of a single object, there is a homeomorphism

$$S^{n-2} * |\overline{\mathcal{P}_1}| * \cdots * |\overline{\mathcal{P}_n}| \cong S^{n-3} * |\overline{\mathcal{P}_1}| * \cdots * |\overline{\mathcal{P}_{i-1}}| * |\overline{\mathcal{P}_{i+1}}| \cdots * |\overline{\mathcal{P}_n}|.$$

2.2. Poset of partitions.

Our main interest is in the poset of partitions (i.e., equivalence relations) of a finite set. For a finite set S, let Part_S be the collection of partitions of S, partially ordered by refinement. We say that $\lambda_1 < \lambda_2$ if λ_2 is finer than λ_1 (this is opposite of the convention we used in some of our previous papers, but it seems convenient here). Throughout the paper, let $\mathbf{n} = \{1, \ldots, n\}$ be the standard set with n elements. We denote the poset of partitions of $\mathbf{n}$ by Part_n.

The poset Part_n has an initial and a final object. As usual we denote them by $\hat{0}$ and $\hat{1}$ (we will not include n in the notation for these, trusting that it will not cause confusion). They may be called the "indiscrete" and the "discrete" partition respectively (note that for $n = 1$, $\hat{0} = \hat{1}$).

We often will want to consider the poset of partitions without the initial and final object, so we introduce separate notation for it. For $n > 1$ define $\Pi_n = \mathrm{Part}_n \setminus \{\hat{0}, \hat{1}\}$. It is well-known that for $n \geq 3$

there is a homotopy equivalence $|\Pi_n| \simeq \bigvee_{(n-1)!} S^{n-3}$. For $n = 1, 2$ this equivalence can not be true as stated, if only because the right hand side is not defined. Note that $\Pi_2 = \emptyset$, a.k.a, S^{-1}, which is compatible with the general formula. In keeping with the conventions of section 2.1, we identify $|\Pi_1|$ with S^{-2}. This means that $|\Pi_1|$ is not defined as a space, but $|\Pi_1|^\diamond = \emptyset$, and for all $n \geq 0$, $S^n * |\Pi_1| = S^n \wedge |\Pi_1|^\diamond := S^{n-1}$.

Let $\widetilde{H}^*(X)$ be the reduced cohomology of a space X. It is defined using the augmented simplicial or singular chain complex. Since for $n \geq 3$ Π_n is a wedge of spheres of dimension $n - 3$, it only has reduced homology in this dimension. For $n = 2$ $\Pi_2 = \emptyset$ and it is still true that $\widetilde{H}^{-1}(\emptyset) \cong \mathbb{Z}$, if one defines cohomology using the augmented chain complex. Since we defined $|\Pi_1|$ to be S^{-2}, we also define $\widetilde{H}^{-2}(\Pi_1) = \mathbb{Z}$. This convention will enable us to make statement valid for all Π_n, without needing to make exceptions for $n = 1$.

We notice that the poset Part_n (and also Π_n) is equipped with a rank function: for a partition λ, let $\mathrm{rk}(\lambda)$ be the number of equivalence classes of λ. We note the following elementary properties of the rank function: if $\lambda_1 < \lambda_2$ then $\mathrm{rk}(\lambda_1) < \mathrm{rk}(\lambda_2)$. In particular, distinct partitions of same rank are not comparable. The action of the symmetric group on Part_n preserves the rank function.

We will need to deal with intervals in Part_n. Let λ_1, λ_2 be partitions, and suppose that $\lambda_1 < \lambda_2$. Let us suppose that λ_1 has k equivalence classes, and label them $c_1, \ldots, c_k$. Since $\lambda_1 < \lambda_2$, each equivalence class of λ_1 is a union of classes of λ_2. Let us suppose that for $i = 1, \ldots, k$, c_i is the union of m_i classes of λ_2. Then it is easy to see the following formula for the closed interval $[\lambda_1, \lambda_2]$

$$[\lambda_1, \lambda_2] \cong \mathrm{Part}_{m_1} \times \cdots \times \mathrm{Part}_{m_k}.$$

It follows that the open interval (λ_1, λ_2) is isomorphic to the poset $\overline{\mathrm{Part}_{m_1} \times \cdots \times \mathrm{Part}_{m_k}}$. By Proposition 2.8, there is a homeomorphism

$$(2.9) \qquad\qquad |(\lambda_1, \lambda_2)| \cong S^{k-2} * |\Pi_{m_1}| * \cdots * |\Pi_{m_k}|$$

Remark 2.10. We will need the following simple observation: if $\lambda_1 < \lambda_2$, and $\mathrm{rk}(\lambda_1) < r < \mathrm{rk}(\lambda_2)$ then there are at least two partitions λ of rank r satisfying $\lambda_1 < \lambda < \lambda_2$. For example, if $\mathrm{rk}(\lambda_2) - \mathrm{rk}(\lambda_1) > 2$ then an intermediate partition will have rank equal to $\mathrm{rk}(\lambda_1) + 1$. There are two possibilities: either λ_2 is obtained from λ_1 by splitting two equivalence classes in two parts each, or by splitting a single class in three parts. In the first case, there are two intermediate partitions between λ_1 and λ_2, in the second case there are three.

2.2.1. *Stars and links.* Let $\mathcal{P}$ be a poset. Let $P = (p_0 < \cdots < p_k)$ be a non-empty increasing chain in $\mathcal{P}$. The star of P is the poset of elements of $\mathcal{P}$ that are comparable with each one of the elements $p_0, \ldots, p_k$. Denote it $\mathrm{St}(P)$. The star decomposes canonically as a join of posets. More precisely

$$\mathrm{St}(P) \cong \mathcal{P}_{<p_0} * \{p_0\} * (p_0, p_1) * \{p_1\} * \cdots * \{p_k\} * \mathcal{P}_{>p_k}.$$

It follows in particular that $|\mathrm{St}(P)|$ is contractible, since the join of any space with a point is contractible.

The link of P, denoted $\mathrm{Lk}(P)$ is the subposet of $\mathrm{St}(P)$ consisting of elements that are distinct from $p_0, \ldots, p_k$. The link also decomposes as a join. More precisely

$$\mathrm{Lk}(P) \cong \mathcal{P}_{<p_0} * (p_0, p_1) * \cdots * \mathcal{P}_{>p_k}.$$

Let us consider the case of Π_n. Let $\Lambda = (\lambda_0 < \cdots < \lambda_k)$ be an increasing chain of partitions in Π_n. Then $|\mathrm{St}(\Lambda)|$ is a subspace of $|\Pi_n|$. We will need to identify the boundary of this subspace.

Lemma 2.11. *The boundary of $|\mathrm{St}(\Lambda)|$ is the geometric realization of the subcomplex of $\mathrm{St}(\Lambda)$ defined by chains that do not contain Λ as a subchain.*

Proof. Let $\Theta = (\theta_0 < \cdots < \theta_m)$ be an increasing chain in $\mathrm{St}(\Lambda)$. It belongs to the boundary precisely if there exists a partition μ that is comparable with all the θ_js, but not all the λ_is. If Λ is a subchain of Θ then clearly such a partition does not exists. Conversely, if Θ does not contain one of the λ_is then by remark 2.10 one can find another partition μ of the same rank as λ_i that is comparable with all the θ_js. Then μ is not comparable with λ_i. $\square$

Definition 2.12. Let us say that an increasing chain of partitions $\Lambda = (\lambda_0 < \cdots < \lambda_k)$ is *binary* if for all $i \geq 0$, each equivalence class of λ_i is the union of at most two equivalence classes of λ_{i+1}. For $i = k$ this means that each component of λ_k has at most two elements. However, note that we do not require that λ_0 has two equivalence classes.

We need to analyze the boundary of $\mathrm{St}(\Lambda)$ for a binary chain. We already observed that

$$\mathrm{St}(\Lambda) = (\hat{0}, \lambda_0) * \{\lambda_0\} * (\lambda_0, \lambda_1) * \cdots * \{\lambda_k\} * (\lambda_k, \hat{1}).$$

By (2.9), $(\hat{0}, \lambda_0) \cong \Pi_c$, where c is the set of equivalence classes of λ_0. The geometric realization of an open interval $(\lambda_{i-1}, \lambda_i)$ is homeomorphic to a join of a sphere and, for each equivalence class of λ_{i-1} a copy of $|\Pi_{m_j}|$, where m_j is number of classes of λ_i within the given class of λ_{i-1}. The assumption that Λ is binary means precisely that $m_j \leq 2$

in all cases. This means that each Π_{m_j} is either S^{-1}, i.e., empty, if $m_j = 2$ or is S^{-2}, if $m = 1$. Recall that taking join of a sphere with S^{-2} has the effect of lowering the dimension of a sphere by one. This means that $|(\lambda_{i-1}, \lambda_i)|$ is homeomorphic to a sphere of dimension $k-2$, where k is the number of equivalence classes of λ_{i-1} that are the union of two classes of λ_i. It follows that $|\{\lambda_0\} * (\lambda_0, \lambda_1) * \cdots * \{\lambda_k\} * (\lambda_k, \hat{1})|$ is homeomorphic to a simplex. Its dimension is easily calculated to be $n - c - 1$. We conclude that there are homeomorphisms

$$(2.13) \qquad |\operatorname{St}(\Lambda)| \cong \operatorname{St}(\Lambda)_{<\lambda_0} * \operatorname{St}(\Lambda)_{\geq \lambda_0} \cong |\Pi_c| * \Delta^{n-c-1}$$

Let $\partial \Delta^{n-c-1}$ be the topological boundary of the simplex. It is home-omorphic to S^{n-c-2} of course. We may identify $|\Pi_c| * \partial \Delta^{n-c-1}$ with a subspace of $|\operatorname{St}(\Lambda)|$. Our next goal is to prove the unsurprising assertion that this is in fact the boundary of $|\operatorname{St}(\Lambda)|$ considered as a subspace of $|\Pi_n|$.

Proposition 2.14. *Using the notation above, the boundary of $|\operatorname{St}(\Lambda)|$ is precisely $|\Pi_c| * \partial \Delta^{n-c-1}$.*

Proof. We already saw that there is a decomposition

$$\operatorname{St}(\Lambda) = \operatorname{St}(\Lambda)_{<\lambda_0} * \operatorname{St}(\Lambda)_{\geq \lambda_0} \cong \Pi_c * \big(\{\lambda_0\} * (\lambda_0, \lambda_1) * \cdots * \{\lambda_k\} * (\lambda_k, \hat{1})\big).$$

By Lemma 2.11, the boundary of $|\operatorname{St}(\Lambda)|$ is given by the subcomplex spanned by all chains that do not contain Λ as a subchain. Let $\operatorname{Bd}(\operatorname{St}(\Lambda)_{\geq \lambda_0})$ be the subcomplex of $\operatorname{St}(\Lambda)_{\geq \lambda_0}$ spanned by chains that do not contain Λ as a subchain. Clearly, the boundary of $|\operatorname{St}(\Lambda)|$ is the geometric realization of $\Pi_c * \operatorname{Bd}(\operatorname{St}(\Lambda)_{\geq \lambda_0})$. So we need to show that $\operatorname{Bd}(\operatorname{St}(\Lambda)_{\geq \lambda_0})$ is in fact the topological boundary of $\operatorname{St}(\Lambda)_{\geq \lambda_0}$.

We saw that $\operatorname{St}(\Lambda)_{\geq \lambda_0}$ is combinatorially equivalent to a simplex. It is well-known, and not hard to prove, that a chain belongs to the topological boundary if and only if the geometric realization of the link of the chain is contractible. So we need to prove that the link of a chain in $\operatorname{St}(\Lambda)_{\geq \lambda_0}$ is contractible precisely if it does not contain Λ as a subchain.

Let $\Theta = (\theta_0 < \cdots < \theta_m)$ be a chain in $\operatorname{St}(\Lambda)_{\geq \lambda_0}$. The link of this chain is the join of intervals

$$[\lambda_0, \theta_0) * (\theta_0, \theta_1) * \cdots * (\theta_m, \hat{1}),$$

where the intervals are taken in $\operatorname{St}(\Lambda)_{\geq \lambda_0}$. If Θ does not contain Λ as a subchain, then one of these intervals contains a λ_i. Let us suppose that $\lambda_i \in (\theta_{j-1}, \theta_j)$. Then $(\theta_{j-1}, \theta_j) = (\theta_{j-1}, \lambda_i) * \{\lambda_i\} * (\lambda_i, \theta_j)$, and so the geometric realization is contractible. Conversely, if Θ does contain Λ as a subchain then the link is a join of copies of Π_m for various m, and therefore is not contractible. $\qquad \square$

2.3. **Equivariant topology.** Let G be a finite group. We only want to consider nice spaces with an action of G. Thus when we say that X is a G-space, we assume that X is homeomorphic, as a G space, to the geometric realization of a G-simplicial set. Equivalently, we could choose to work with G-CW complexes.

A (finite, pointed) G space X can be written as a (finite, pointed) homotopy colimit of spaces of the form G/H, where H is an isotropy group of X.

A G-map $f \colon X \to Y$ is a G-equivariant weak equivalence if it induces weak equivalences of fixed point spaces $f^H \colon X^H \to Y^H$ for every subgroup H of G. It is well-known that for nice spaces, an equivariant weak equivalence is an equivariant homotopy equivalence.

In fact, it is sometimes not necessary to check the fixed points condition for every subgroup of G. It is enough to do it for subgroups that occur as an isotropy of one of the spaces in questions. A proof of the following lemma can be found in [10, 4.1], but it is older than that.

Lemma 2.15. *Let $f \colon X \to Y$ be a G-map. Suppose that f induces an equivalence $f^H \colon X^H \to Y^H$ for every subgroup H that occurs as an isotropy of X or of Y. Then f is an equivariant equivalence.*

Suppose K, H are subgroups of G and X is a pointed space with an action of H. Then one may form the G-space $G_+ \wedge_H X$. We will need several times to calculate the fixed points space $(G_+ \wedge_H X)^K$. The following lemma is proved by a standard calculation with the double cosets formula.

Lemma 2.16. *Let $N_G(K; H) = \{g \in G \mid g^{-1}Kg \subset H\}$. Note that H acts on this set on the right. There is a homeomorphism*

$$(G_+ \wedge_H X)^K \cong \bigvee_{[g] \in N_G(K;H)/H} X^{g^{-1}Kg}.$$

In particular, if K is not subconjugate to H then $(G_+ \wedge_H X)^K \cong *$.

We will often have to deal with the following special case: $G \subset \Sigma_d \subset \Sigma_n$, where $d|n$ and Σ_d is embedded diagonally in Σ_n (we hope that the reader will not be confused by the change in the role of G).

Lemma 2.17. *Suppose $G \subset \Sigma_d \subset \Sigma_n$, where Σ_d is embedded in Σ_n diagonally. Let σ be an element of $N_{\Sigma_n}(G; \Sigma_d)$. Thus σ is an element of Σ_n that conjugates G to a subgroup of Σ_d. Then there exists an element $\eta \in \Sigma_d$ such that conjugation by η on G is the same as conjugation by σ.*

Proof. An inclusion of G into Σ_n is the same thing as an effective action of G on $\mathbf{n}$. If the inclusion factors through Σ_d, then $\mathbf{n}$ is isomorphic,

as a G-set, to a disjoint union of $\frac{n}{d}$ copies of the set $\mathbf{d}$, which equipped with an effective action of G. Specifying an element σ of Σ_n that conjugates G into a subgroup of Σ_d is the same as specifying a second action of G on $\mathbf{d}$ (let $\mathbf{d}'$ denote $\mathbf{d}$ with the second action of G), a group isomorphism $\sigma_G \colon G \to G$, and an isomorphism of sets

$$\sigma \colon \coprod_{\frac{n}{d}} \mathbf{d} \to \coprod_{\frac{n}{d}} \mathbf{d}'$$

that is equivariant with respect to the isomorphism σ_G. Such an isomorphism exists if and only if $\coprod_{\frac{n}{d}} \mathbf{d}$ and $\coprod_{\frac{n}{d}} \mathbf{d}'$ are isomorphic permutation representations of G. But this is possible if and only if $\mathbf{d}$ and $\mathbf{d}'$ are isomorphic permutation representations of G, which means that the isomorphism σ_G can be realized as conjugation by an element of Σ_d. $\qquad\square$

Let $C_W(G)$ denote the centralizer of G in W.

Corollary 2.18. *Suppose $G \subset \Sigma_d \subset W \subset \Sigma_n$ where Σ_d is embedded diagonally in Σ_n, and W is any intermediate subgroup. There is an isomorphism of sets, induced by inclusion*

$$C_W(G)/C_{\Sigma_d}(G) \xrightarrow{\cong} N_W(G;\Sigma_d)/\Sigma_d.$$

Proof. By definition,

$$N_W(G;\Sigma_d) = \{w \in W \mid w^{-1}Gw \subset \Sigma_d\}.$$

By Lemma 2.17 for every such w one can find an $\eta \in \Sigma_d$ such that conjugation by η on G coincides with conjugation by w. It follows that $w\eta^{-1} \in C_W(G)$. Thus ever element of $N_W(G;\Sigma_d)/\Sigma_d$ has a representative in $C_W(G)$. Clearly two elements of $C_W(G)$ differ by an element of Σ_d if and only if they differ by an element of $C_{\Sigma_d}(G)$. The corollary follows. $\qquad\square$

Corollary 2.19. *Suppose $G \subset \Sigma_d \subset W \subset \Sigma_n$ where Σ_d is embedded diagonally in Σ_n, and W is any intermediate subgroup. Suppose X is a space with an action of Σ_d. Then there is a homeomorphism*

$$(W_+ \wedge_{\Sigma_d} X)^G \cong C_W(G)_+ \wedge_{C_{\Sigma_d}(G)} X^G.$$

In particular, if X^G is contractible, then so is $(W_+ \wedge_{\Sigma_d} X)^G$.

Proof. By Lemma 2.16 there is a homeomorphism

$$(W_+ \wedge_{\Sigma_d} X)^G \cong \bigvee_{N_W(G;\Sigma_d)/\Sigma_d} X^{w^{-1}Gw}.$$

By Corollary 2.18, $N_W(G; \Sigma_d)/\Sigma_d \cong C_W(G)/C_{\Sigma_d}(G)$. In particular, w can always be chosen in the centralizer of G, so $X^{w^{-1}Gw} = X^G$. It follows that

$$(W_+ \wedge_{\Sigma_d} X)^G \cong \bigvee_{C_W(G)/C_{\Sigma_d}(G)} X^G.$$

The right hand side is isomorphic to $C_W(G)_+ \wedge_{C_{\Sigma_d}(G)} X^G$ but the latter presentation is better in the sense that it gives the correct action of $C_W(G)$. $\qquad\square$

3. FREE LIE ALGEBRAS AND THE ALGEBRAIC HILTON-MILNOR THEOREM

For a free abelian group V, let $\mathcal{L}[V]$ be the free Lie algebra (over $\mathbb{Z}$) generated by V. If V has a $\mathbb{Z}$-basis $x_1, \ldots, x_k$, we denote $\mathcal{L}[V]$ by $\mathcal{L}[x_1, \ldots, x_k]$.

The multiplication in $\mathcal{L}[V]$ satisfies the following well-known relations: the square zero relation $[u, u] = 0$ (which implies anticommutativity) and the Jacobi relation $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$. In fact, $\mathcal{L}[x_1, \ldots, x_k]$ can be constructed as the quotient of the free non-associative and non-unital algebra on $\{x_1, \ldots, x_k\}$ by the two sided ideal generated by these relations (for example see [22], Theorem 0.4 and its proof).

Throughout the paper, monomials are understood to be in non-commutative and non-associative variables. We will also refer to them as parenthesized monomials. Elements of $\mathcal{L}[x_1, \ldots, x_k]$ can be represented as linear combinations of parenthesized monomials in $x_1, \ldots, x_k$. As is customary in the context of Lie algebras, we will use square brackets to indicate the parenthesization. For example, $[[x_1, [x_2, x_1]], x_2]$ represents an element of $\mathcal{L}[x_1, x_2]$.

Let $(n_1, \ldots, n_k)$ be non-negative integers. We say that a monomial in $x_1, \ldots, x_k$ has multi-degree $(n_1, \ldots, n_k)$ if it contains n_i copies of x_i, for all $i = 1, \ldots, k$. For example, the monomial $[[x_1, [x_2, x_1]], x_2]$ has multi-degree $(2, 2)$. We define $M(n_1, \ldots, n_k) \subset \mathcal{L}[x_1, \ldots, x_k]$ to be the $\mathbb{Z}$-submodule generated by monomials of multi-degree $(n_1, \ldots, n_k)$. Note that $M(0, \ldots, 0)$ is the trivial group.

The following theorem is well-known.

Theorem 3.1. *Additively, $\mathcal{L}[x_1, \ldots, x_k]$ is a free abelian group. Moreover, there is an additive isomorphism*

$$(3.2) \qquad \mathcal{L}[x_1, \ldots, x_k] \cong \bigoplus_{n_1, \ldots, n_k \geq 0} M(n_1, \ldots, n_k)$$

Proof. For the first assertion see, for example, [22, Corollary 0.10]. The second assertion follows from the first and the fact that the square zero relation and the Jacobi relations are generated by relations that are sums of monomials of the same degree. □

Note that if $u \in M(n_1, \ldots, n_k)$ and $v \in M(m_1, \ldots, m_k)$ then $[u, v]$ and $[v, u]$ are in $M(n_1 + m_1, \ldots, n_k + m_k)$. It follows that isomorphism (3.2) makes $\mathcal{L}[x_1, \ldots, x_k]$ into an $\mathbb{N}^k$-graded algebra (here $\mathbb{N}$ stands for non-negative integers). It also has an $\mathbb{N}$-grading, given by the total number of generators in a monomial.

It follows from Theorem 3.1 that each $M(n_1, \ldots, n_k)$ is a free Abelian group. Next, we choose an explicit bases for these groups. There are many known bases for the free Lie algebra. We will use the 'basic monomials' defined by M. Hall and P. Hall. We will now recall the construction.

Definition 3.3. Basic monomials in $x_1, ..., x_k$ and their weights are defined inductively as follows: To begin with, $x_1, \ldots, x_k$ are basic monomials of weight one. Next, suppose we have defined all the basic monomials of weight less than n, and ordered them in some way, so that monomials of lower weight precede those of higher weight. Let us say that the basic monomials of weight less than n are $u_1, \ldots, u_\alpha$, for some $\alpha \geq k$. Then basic monomials of weight n are all monomials of the form $[u_i, u_j]$ such that

    (1) u_i and u_j are basic monomials of weight less than n, and the sum of their weights equals n,
    (2) $i > j$
    (3) either $u_i = x_i$ for some $i \leq k$, or $u_i = [u_s, u_t]$ and $t \leq j$.

Now order the newly formed basic monomial of weight n in some way, append them to the list of basic monomials, and repeat the procedure.

It is well-known that basic monomials form a basis for $\mathcal{L}[x_1, \ldots, x_k]$ [12, 13]. Let $B(n_1, \ldots, n_k)$ be the set of basic monomials of multi-degree $(n_1, \ldots, n_k)$. Then $B(n_1, \ldots, n_k)$ is a basis for $M(n_1, \ldots, n_k)$. The size of $B(n_1, \ldots, n_k)$ is given by the Witt formula (1.1).

Last topic in this section is the algebraic Hilton-Milnor theorem. In topology, the Hilton-Milnor theorem is a statement about the homotopy decomposition of the space $\Omega\Sigma(X_1 \vee \ldots \vee X_k)$ as a weak product indexed by basic monomials in k variables $x_1, \ldots, x_k$. Equivalently, one can view the theorem as a statement about the homotopy decomposition of the pointed free topological group generated by $X_1, \ldots, X_k$, where X_i are connected spaces. This is so because the functor $\Omega\Sigma$

is equivalent to the pointed free topological group functor. By algebraic Hilton-Milnor theorem we mean an analogous decomposition of a free Lie algebra on a direct sum. Let us point out a couple of differences between the topological and algebraic versions of the theorem: the algebraic version gives an isomorphism, rather than a homotopy equivalence, and it does not require a connectivity hypothesis.

Before we state the theorem, we need to define a certain operation on monomials.

Definition 3.4. Let $w \in B(n_1, \ldots, n_k)$ (more generally, w can be any monomial of multi-degree $(n_1, \ldots, n_k)$). By definition, w is a monomial in $x_1, \ldots, x_k$ that contains n_i copies of x_i for each i. The *resolution* of w, denoted $\tilde{w}$, is the multi-linear monomial in n variables obtained from w by replacing the n_1 occurrences of x_1 with $x_1, \ldots, x_{n_1}$, in the order in which they occur in w, replacing the n_2 occurrences of x_2 with $x_{n_1+1}, \ldots, x_{n_1+n_2}$, and so on.

Example 3.5. Consider the following monomial of multi-degree $(2, 2)$ $w(x_1, x_2) = [[[x_2, x_1], x_1], x_2]$. To obtain the resolution of w replace the two occurrences of x_1 with x_1 and x_2, and the two occurrences of x_2 with x_3 and x_4. We obtain the monomial $\tilde{w}(x_1, x_2, x_3, x_4) = [[[x_3, x_1], x_2], x_4]$ in Lie_4.

Now let $V_1, \ldots, V_k$ be free Abelian groups. Let $w \in B(n_1, \ldots, n_k)$ be a basic monomial. We associate with w a group homomorphism

$$\alpha_w \colon V_1^{\otimes n_1} \otimes \cdots \otimes V_k^{\otimes n_k} \to \mathcal{L}[V_1 \oplus \cdots \oplus V_k].$$

The homomorphism is defined by the following formula

$$\alpha_w \left(\otimes_{i=1}^k \otimes_{j=1}^{n_i} v_i^j \right) = \tilde{w}(v_1^1, v_1^2, \ldots, v_1^{n_1}, v_2^1, \ldots, v_k^{n_k}).$$

where $v_i^j \in V_i$ for all $1 \leq i \leq k, 1 \leq j \leq n_i$. Since $\tilde{w}$ is a multi-linear monomial in the correct number of variables, this is a well-defined homomorphism.

Example 3.6. Let $w = [[[x_2, x_1], x_1], x_2]$. Then w determines a homomorphism $\alpha_w \colon V_1^{\otimes 2} \otimes V_2^{\otimes 2} \to \mathcal{L}[V_1 \oplus V_2]$, by the formula

$$\alpha_w(v_1^1 \otimes v_1^2 \otimes v_2^1 \otimes v_2^2) = \tilde{w}(v_1^1, v_1^2, v_2^1, v_2^2) = [[[v_3, v_1], v_2], v_4].$$

The homomorphism α_w extends to a homomorphism of Lie algebras

$$\tilde{\alpha}_w \colon \mathcal{L}[V_1^{\otimes n_1} \otimes \cdots \otimes V_k^{\otimes n_k}] \to \mathcal{L}[V_1 \oplus \cdots \oplus V_k].$$

Taking direct sum over $w \in B(n_1, \ldots, n_k)$ we obtain a homomorphism of abelian groups

$$\Sigma \tilde{\alpha}_w \colon \bigoplus_{w \in B(n_1, \ldots, n_k)} \mathcal{L}[V_1^{\otimes n_1} \otimes \cdots \otimes V_k^{\otimes n_k}] \to \mathcal{L}[V_1 \oplus \cdots \oplus V_k].$$

Remark 3.7. There is an isomorphism

$$\bigoplus_{w \in B(n_1,\ldots,n_k)} \mathcal{L}[V_1^{\otimes n_1} \otimes \cdots \otimes V_k^{\otimes n_k}] \cong M(n_1,\ldots,n_k) \otimes \mathcal{L}[V_1^{\otimes n_1} \otimes \cdots \otimes V_k^{\otimes n_k}]$$

so we constructed a homomorphism

$$M(n_1,\ldots,n_k) \otimes \mathcal{L}[V_1^{\otimes n_1} \otimes \cdots \otimes V_k^{\otimes n_k}] \to \mathcal{L}[V_1 \oplus \cdots \oplus V_k].$$

However, we emphasize that this isomorphism is not canonical - it depends on a choice of basis of $M(n_1,\ldots,n_k)$.

Finally, taking direct sum over all n-tuples $(n_1,\ldots,n_k)$, we obtain a homomorphism of abelian groups.

$$(3.8) \qquad \bigoplus_{n_1,\ldots,n_k \geq 0} \bigoplus_{B(n_1,\ldots,n_k)} \mathcal{L}[V_1^{\otimes n_1} \otimes \cdots \otimes V_k^{\otimes n_k}] \to \mathcal{L}[V_1 \oplus \cdots \oplus V_k].$$

This is not a homomorphism of Lie algebras, but notice that it is a homomorphism of graded abelian groups, provided that on the left hand side one endows $V_1^{\otimes n_1} \otimes \cdots \otimes V_k^{\otimes n_k}$ with grading $n_1 + \cdots + n_k$.

The following is the algebraic Hilton-Milnor theorem

Theorem 3.9. *The homomorphism* (3.8) *is an isomorphism.*

Proof. Let V, W be free Abelian groups. Consider the natural split surjection of Lie algebras. $\mathcal{L}[V \oplus W] \to \mathcal{L}[V]$. The key to the proof is the following lemma, which identifies the kernel of this homomorphism as a free Lie algebra with an explicit set of generators.

Lemma 3.10. *There is a short exact sequence of free Lie algebras, (which splits on the level of graded abelian groups).*

$$0 \to \mathcal{L}\left[\bigoplus_{i=0}^{\infty} W \otimes V^{\otimes i}\right] \to \mathcal{L}[V \oplus W] \to \mathcal{L}[V] \to 0.$$

Here the first map is determined by the group homomorphisms $W \otimes V^{\otimes i} \to \mathcal{L}[V \oplus W]$ that send the element $w \otimes v_1 \otimes \cdots \otimes v_i$ to the iterated commutator $[...[w, v_1], v_2], \ldots]v_i]$.

The lemma is proved as part of Example 8.7.4 ("The Hilton-Milnor example") in Neisendorfer's book [20]. It should be noted that Neisendorfer works with graded Lie algebras, and in the proof he uses some lemmas that assume that Lie algebras are zero-connected. However, Lemma 3.10 holds for ungraded Lie algebras as well. To see this, notice that we can make our algebras connected by assigning V and W grading 2, and this does not change anything. Whether V has grading 0 or 2, the resulting Lie algebras $L[V]$ are canonically isomorphic as graded abelian groups.

Once we have Lemma 3.10, the rest of the proof of the theorem proceeds exactly as for the classic Hilton-Milnor theorem. For details, see the proof of Theorem 4 in [18]. The input for that theorem is the exact analogue of Lemma 3.10 for topological free groups. The proof of our theorem can emulate Milnor's proof almost word for word. One just needs to replace "free group" with "free Lie algebra", "homotopy equivalence" with "isomorphism" and "complexes" with "free Abelian groups". $\qquad \square$

4. From Lie algebras to the Lie representation

Recall from previous section that $M(1, \ldots, 1) \subset \mathcal{L}[x_1, \ldots, x_n]$ is the submodule generated by monomials that contain each generator exactly once. We call such monomials multilinear. By the Witt formula, $M(1, \ldots, 1)$ is a free abelian group of rank $(n-1)!$. Moreover, there is a natural action of Σ_n, permuting the generators. We arrange it so that it is a right action. We call $M(1, \ldots, 1)$, considered as an integral representation of Σ_n, the Lie representation, and denote it by Lie_n

The sequence of representations $\mathrm{Lie}_1, \mathrm{Lie}_2, \ldots, \mathrm{Lie}_n, \ldots$ forms an operad in abelian groups - the Lie operad. Let us recall what this means. Suppose that we have an isomorphism $\phi \colon \mathbf{n_1} \coprod \cdots \coprod \mathbf{n_i} \overset{\cong}{\to} \mathbf{n}$. Then there is a homomorphism associated with ϕ,

$$s_\phi \colon \mathrm{Lie}_i \otimes \mathrm{Lie}_{n_1} \otimes \cdots \otimes \mathrm{Lie}_{n_i} \to \mathrm{Lie}_n \, .$$

The homomorphism s_ϕ is natural with respect to isomorphisms of the sets $\mathbf{n_1}, \ldots, \mathbf{n_i}$ and also with respect to permutations of these set. Let us describe s_ϕ explicitly in terms of monomials. Recall that Lie_n is generated by (parenthesized) multilinear monomials in variables $x_1, \ldots, x_n$. Let

$$w_i(x_1, \ldots, x_i), w_{n_1}(x_1, \ldots, x_{n_1}), \ldots, w_{n_i}(x_1, \ldots, x_{n_i})$$

be monomials representing elements of $\mathrm{Lie}_i, \mathrm{Lie}_{n_1}, \ldots, \mathrm{Lie}_{n_i}$ respectively. To distinguish between elements of $\mathbf{n_1}, \mathbf{n_2}$, etc., let $1^j, \ldots n_j{}^j$ be elements of $\mathbf{n_j}$. Then $s_\phi(w_i \otimes w_{n_1}, \ldots, w_{n_i})(x_1, \ldots, x_n)$ equals to

$$w_i(w_{n_1}(x_{\phi(1^1)}, \ldots, x_{\phi(n_1{}^1)}), \ldots, w_{n_i}(x_{\phi(1^i)}, \ldots, x_{\phi(n_i{}^i)})).$$

In words, $s_\phi(w_i \otimes w_{n_1}, \ldots, w_{n_i})$ is obtained by taking the monomial $w_i(x_1, \ldots, x_i)$, substituting the monomials $w_{n_1}, \ldots, w_{n_i}$ for the variables $x_1, \ldots, x_i$, and then reindexing the variables as prescribed by the isomorphism ϕ.

Unless said otherwise, we take ϕ to be the obvious isomorphism that takes the elements of $\mathbf{n_1}$ to $1, \ldots, n_1$, the elements of $\mathbf{n_2}$ to $n_1 + 1, \ldots, n_1 + n_2$, etc.

Lie algebras are the same thing as algebras over the Lie operad. There is a well-known formula for a free algebra over an operad. In the case of Lie algebras it says that there is a natural isomorphism

$$(4.1) \qquad \epsilon \colon \bigoplus_{n=1}^{\infty} \mathrm{Lie}_n \otimes_{\mathbb{Z}[\Sigma_n]} V^{\otimes n} \to \mathcal{L}[V].$$

See [17], Propositions 1.25 and 1.27. The isomorphism ϵ is described explicitly in terms of monomials as follows. Let $w(x_1, \ldots, x_n)$ be a parenthesized multilinear monomial in $x_1, \ldots x_n$. The group $V^{\otimes n}$ is generated by elements of the form $v_1 \otimes \cdots \otimes v_n$. The homomorphism ϵ sends $w \otimes v_1 \otimes \cdots \otimes v_n$ to $w(v_1, \ldots, v_n)$.

Note that ϵ preserves the natural grading. Thus ϵ restricts to an isomorphism of $\mathrm{Lie}_n \otimes_{\mathbb{Z}[\Sigma_n]} V^{\otimes n}$ with the degree n part of $\mathcal{L}[V]$.

The isomorphism ϵ, together with the algebraic Hilton-Milnor theorem, yield a branching rule for the restriction of Lie_n to a Young subgroup of Σ_n. To see this, consider $\mathcal{L}[V_1 \oplus \cdots \oplus V_k]$, the free Lie algebra on a direct sum of k free abelian groups. Combining the operadic model for the free Lie algebra and the Hilton-Milnor theorem, we obtain a commutative square of isomorphisms

(4.2)

$$\bigoplus_{\substack{n_1,\ldots,n_k \geq 0 \\ w \in B(n_1,\ldots,n_k) \\ d \geq 1}} \mathrm{Lie}_d \otimes_{\Sigma_d} (V_1^{\otimes n_1} \otimes \ldots \otimes V_k^{\otimes n_k})^{\otimes d} \quad \to \quad \bigoplus_{n \geq 1} \mathrm{Lie}_n \otimes_{\Sigma_n} (V_1 \oplus \ldots \oplus V_k)^{\otimes n}$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\bigoplus_{\substack{n_1,\ldots,n_k \geq 0 \\ w \in B(n_1,\ldots,n_k)}} \mathcal{L}[V_1^{\otimes n_1} \otimes \cdots \otimes V_k^{\otimes n_k}] \qquad \to \qquad \mathcal{L}[V_1 \oplus \cdots \oplus V_k]$$

Here the top map is defined to be the unique isomorphism that makes the diagram commute. An easy diagram chase shows that this isomorphism preserves multi-degree in the variables $V_1, \ldots, V_k$. This means that both the source and the target of the top map are direct sums of terms of the form $A \otimes_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} V_1^{\otimes n_1} \otimes \cdots V_k^{\otimes n_k}$, where $n_1, \ldots, n_k$ are natural numbers and A is a (free abelian) group with an action of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$. Fixing a k-tuple $(n_1, \ldots, n_k)$ and restricting to terms of multi-degree $(n_1, \ldots, n_k)$, we find that the top isomorphism restricts to an isomorphism (natural in the variables $V_1, \ldots, V_k$) of the following form:

$$\bigoplus_{\substack{d \mid \gcd(n_1,\ldots,n_k) \\ w \in B(\frac{n_1}{d},\ldots,\frac{n_k}{d})}} \mathrm{Lie}_d \otimes_{\mathbb{Z}[\Sigma_d]} (V_1^{\otimes \frac{n_1}{d}} \otimes \ldots \otimes V_k^{\otimes \frac{n_k}{d}})^{\otimes d} \to$$

$$\to \mathrm{Lie}_n \otimes_{\mathbb{Z}[\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}]} (V_1^{\otimes n_1} \otimes \ldots \otimes V_k^{\otimes n_k}).$$

Recall that for each d that divides $n_1, \ldots, n_k$, we view Σ_d as a subgroup of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$ via the diagonal inclusion. It follows that the source of this isomorphism can be rewritten as follows

$$\bigoplus_{\substack{d \mid \gcd(n_1, \ldots, n_k) \\ w \in B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})}} \mathrm{Lie}_d \otimes_{\mathbb{Z}[\Sigma_d]} \mathbb{Z}[\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}] \otimes_{\mathbb{Z}[\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}]} (V_1^{\otimes n_1} \otimes \ldots \otimes V_k^{\otimes n_k}).$$

Any natural isomorphism between this functor and the functor

$$\mathrm{Lie}_n \otimes_{\mathbb{Z}[\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}]} (V_1^{\otimes n_1} \otimes \ldots \otimes V_k^{\otimes n_k})$$

is induced by an isomorphism of $\mathbb{Z}[\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}]$-modules

$$(4.3) \qquad \bigoplus_{\substack{d \mid \gcd(n_1, \ldots, n_k) \\ w \in B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})}} \mathrm{Lie}_d \otimes_{\mathbb{Z}[\Sigma_d]} \mathbb{Z}[\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}] \xrightarrow{\cong} \mathrm{Lie}_n.$$

(One way to see this is to take $(n_1, \ldots, n_k)$-cross-effects of these functors of $V_1, \ldots, V_k$).

Recall that the isomorphism is induced by diagram (4.2). Our next goal is to describe the homorphism (4.3) explicitly. To do this it is enough to describe, for each d that divides $n_1, \ldots, n_k$, a Σ_d-equivariant homomorphism

$$(4.4) \qquad \bigoplus_{B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})} \mathrm{Lie}_d \to \mathrm{Lie}_n.$$

Let $w \in B(n_1, \ldots, n_k)$. In Definition 3.4 we defined the resolution of w, denoted $\tilde{w}$, to be a certain multi-linear monomial in n variables. In particular, resolution defines a function

$$B(n_1, \ldots, n_k) \to \mathrm{Lie}_n.$$

We emphasize that $B(n_1, \ldots, n_k)$ is just a set, so this is just a function between sets. Sometimes it is convenient to think of it as a homomorphism $\mathbb{Z}[B(n_1, \ldots, n_k)] \to \mathrm{Lie}_n$. In a similar way, we have a homomorphism $\mathbb{Z}[B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})] \to \mathrm{Lie}_{\frac{n}{d}}$. Now consider the following sequence of homomorphisms

$$\mathrm{Lie}_d \otimes \mathbb{Z}\left[B\left(\frac{n_1}{d}, \ldots, \frac{n_k}{d}\right)\right] \to \mathrm{Lie}_d \otimes \mathbb{Z}\left[B\left(\frac{n_1}{d}, \ldots, \frac{n_k}{d}\right)^d\right] \xrightarrow{\cong}$$

$$\to \mathrm{Lie}_d \otimes \mathbb{Z}\left[B\left(\frac{n_1}{d}, \ldots, \frac{n_k}{d}\right)\right]^{\otimes d} \to \mathrm{Lie}_d \otimes \mathrm{Lie}_{\frac{n}{d}}^{\otimes d} \to \mathrm{Lie}_n.$$

Here the first homomorphism is induced by the diagonal $B\left(\frac{n_1}{d}, \ldots, \frac{n_k}{d}\right) \to B\left(\frac{n_1}{d}, \ldots, \frac{n_k}{d}\right)^d$, the second is the obvious isomorphism, the third is induced by the map $B(\frac{n_1}{d}, \ldots, \frac{n_k}{d}) \to \mathrm{Lie}_{\frac{n}{d}}$ that we defined earlier, and

the last one uses the operad structure on $\{\text{Lie}_n\}$. Since the source of this homomorphism is canonically isomorphic to $\bigoplus_{B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})} \text{Lie}_d$, we have constructed a homomorphism as promised in (4.4). The homomorphism is clearly Σ_d-equivariant.

In words, the homomorphism (4.4) can be described as follows. Let $w \in B\left(\frac{n_1}{d}, \ldots, \frac{n_k}{d}\right)$. In particular, w is a monomial of multi-degree $(\frac{n_1}{d}, \ldots, \frac{n_k}{d})$. Starting with w define the monomial $\tilde{w}$ in $\frac{n}{d}$ distinct variables by replacing the $\frac{n_1}{d}$ occurrences of x_1 with $x_1, \ldots, x_{\frac{n_1}{d}}$ in the same order as they occur in w, replacing the $\frac{n_2}{d}$ occurrences of x_2 with $x_{\frac{n_1}{d}+1}, \ldots, x_{\frac{n_1}{d}+\frac{n_2}{d}}$, etc. Then define a homomorphism $\text{Lie}_d \to \text{Lie}_n$ in the following way: given a monomial $w_d(x_1, \ldots, x_d)$, send it to the monomial

(4.5)
$$w_d\big(\tilde{w}(x_1, x_{1+d}, \ldots, x_{1+n-d}), \tilde{w}(x_2, x_{2+d}, \ldots, x_{2+n-d}), \ldots, \tilde{w}(x_d, x_{2d}, \ldots, x_n)\big)$$

Example 4.6. Let $n_1 = n_2 = 4$, $d = 2$. Then (4.4) specializes to a homomorphism $\bigoplus_{B(2,2)} \text{Lie}_2 \to \text{Lie}_8$. The set $B(2,2)$ has one element: the basic monomial $w = [[[x_2, x_1], x_1], x_2]$. Then $\tilde{w}$ is obtained from w by replacing the two occurrences of x_1 with x_1 and x_2, and the two occurrences of x_2 with x_3 and x_4. Thus $\tilde{w}(x_1, x_2, x_3, x_4) = [[[x_3, x_1], x_2], x_4]$. Now the homomorphism $\text{Lie}_2 \to \text{Lie}_8$ is defined follows. Suppose $w_2(x_1, x_2)$ is a monomial in x_1, x_2, representing an element of Lie_2. then w_2 is sent to the following monomial in 8 variables:

$$w_2(\tilde{w}(x_1, x_3, x_5, x_7), \tilde{w}(x_2, x_4, x_6, x_8)).$$

In particular our homomorphism does the following

$$[x_2, x_1] \mapsto [[[[x_6, x_2], x_4], x_8], [[[x_5, x_1], x_3], x_7]]$$
$$[x_1, x_2] \mapsto [[[[x_5, x_1], x_3], x_7], [[[x_6, x_2], x_4], x_8]].$$

Clearly, the homomorphism is Σ_2-equivariant (remember that Σ_2 is embedded in Σ_8 as the subgroup generated by the cycle $(1,2)(3,4)(5,6)(7,8)$).

The homomorphism (4.4) is Σ_d equivariant. Since Σ_d is a subgroup of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$, we can induce it up to a $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-equivariant homomorphism. Taking direct sum over all positive d that divide $n_1, \ldots, n_k$, we obtain the homomorphism promised in (4.3).

A routine diagram chase, using our explicit descriptions of the Hilton-Milnor isomorphism and the isomorphism (4.1), establishes the following proposition.

Proposition 4.7. *The homomorphism* (4.3) *that we just described is the isomorphism induced by the top map in diagram* (4.2).

5. FROM THE LIE REPRESENTATION TO THE PARTITION COMPLEX

It is well-known that there is a close connection between the Lie representation and the (co)homology of the partition complex. More precisely, there is an isomorphism of integral Σ_n representations

$$\mathrm{Lie}_n \cong \widetilde{\mathrm{H}}^{n-3}(\Pi_n)^{\pm}.$$

The superscript $\pm$ indicates tensoring with the sign representation. This isomorphism was first proved by H. Barcelo [6]. Our treatment of this subject was influenced by the paper of Wachs [23], in which Barcelo's isomorphism was generalized, and perhaps simplified.

We will now describe Barcelo's isomorphism, up to a sign indeterminacy. The nerve of Π_n is an $n-3$-dimensional simplicial complex. Let us identify top-dimensional simplices of $|\Pi_n|$ with increasing chain of partitions of $\mathbf{n}$ of the form $\lambda_1 < \lambda_1 < \cdots < \lambda_{n-2}$. Since this is a maximal sequence of partitions, each λ_{i+1} is obtained from λ_i by dividing one of the equivalence classes of λ_i in two parts. For $i = 1, \ldots, n-2$, λ_i has $i+1$ equivalence classes.

We work with simplicial homology and cohomology. A top-dimensional simplex of $|\Pi_n|$ represents an element of $\mathrm{H}^{n-3}(|\Pi_n|)$ via the cochain (which is automatically a cocycle) that takes value 1 on this simplex and is zero otherwise.

Later in the paper we will need the following simple lemma.

Lemma 5.1. *Every top-dimensional simplex of $|\Pi_n|$ represents a non-divisible (in particular non-zero) element of $\mathrm{H}^{n-3}(|\Pi_n|)$.*

Proof. Let Λ be a top-dimensional simplex of $|\Pi_n|$, and let c_Λ be the cochain represented by Λ. It is enough to show that there exists an $n-3$-dimensional simplicial cycle σ of $|\Pi_n|$ such that $\langle c_\Lambda, \sigma \rangle = \pm 1$. Such a cycle σ is an integral linear combination of top-dimensional simplices, in which Λ appears with coefficient ± 1.

We will now describe a way to construct top-dimensional cycles of $|\Pi_n|$. We learned it from the paper of Wachs [23]. Let T be a tree with vertex set $\{1, \ldots, n\}$. Let E be the set of edges of T. This is a set of size $n-1$. A subset $U \subset E$ determines a partition of $\{1, \ldots, n\}$ - the connected components of the subgraph of T with vertex set $\{1, \ldots, n\}$ and edge set U. The empty set and E correspond to the discrete and the indiscrete partition respectively. Let $\overline{\mathcal{B}}_E$ the poset of proper non-empty subsets of E. We have an inclusion of posets from the opposite of $\overline{\mathcal{B}}_E$ into Π_n. We saw in Example 2.2 that the geometric realization of $\overline{\mathcal{B}}_E$, and therefore also of its opposite, is homeomorphic to S^{n-3}. Passing to geometric realizations, we obtain a map $S^{n-3} \to |\Pi_n|$. Choose a

fundamental simplicial cycle in dimension $n - 3$ of $|\overline{\mathcal{B}}_E|$. Its a linear combination of maximal simplices of $|\overline{\mathcal{B}}_E|$, where each simplex appears with coefficient ± 1. It determines a cycle of $|\Pi_n|$. Thus we associated with every tree T with vertex set $\{1, \ldots, n\}$ a simplicial cycle of Π_n. By construction, this cycle is a linear combination of maximal simplices of $|\Pi_n|$, where each simplex occurs with coefficient ± 1 or 0.

It remains to show that for every maximal simplex Λ of $|\Pi_n|$, one can find a tree T_Λ, such that Λ occurs in the cycle associated with T_Λ with a non-zero coefficient. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_{n-2}$ be the chain of partitions comprising Λ. Let us augment Λ by adding the indiscrete and the discrete partition, to obtain the chain $\hat{0} = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n-2} < \lambda_{n-1} = \hat{1}$. Each λ_i has $i+1$ equivalence classes, and each λ_{i+1} is obtained from λ_i by breaking up one equivalence class in two. Define the tree T_Λ as follows: the vertex set is $\{1, \ldots, n\}$. For each i, let S_i be the equivalence class of λ_i that is broken in two parts in λ_{i+1}. Let the two parts of S_i by U_i and U_i'. Then for each $i = 0, \ldots, n-2$, let e_{i+1} be the edge connecting the smallest element of U_i with the smallest element of U_i' (in fact, one can connect any element of U_i with any element of U_i'). Let T_Λ be the graph with vertex set $\{1, \ldots, n\}$ and edge set $e_1, \ldots, e_{n-1}$. Thus T_Λ has $n - 1$ edges. We claim that T_Λ is connected. We want to prove that any two vertices of T_Λ are connected by a path. We will prove that if s and t are in the same class of λ_i for some i then they are connected by a path. We argue by downward induction on i. For $i = n - 1$, λ_i is the discrete partition, so it is obvious. Suppose it is true for $i + 1$. Suppose s and t are in the same class of λ_i. If they are in the same class of λ_{i+1} then they are connected by a path, by induction assumption. Suppose they are not in the same class of λ_{i+1}. Then necessarily one of them is in U_i and the other in U_i'. By construction, there is an edge connecting an element of U_i with an element of U_i'. Thus there is a path connecting s and t. It follows that if s and t are in the same component of λ_0 then they are in the same path. But λ_0 is the indiscrete partition so any two vertices of T_Λ are connected by a path. Since T_Λ is a connected graph with n vertices and $n - 1$ edges, it must be a tree.

It remains to show that Λ occurs in the cycle associated with T_Λ. This is equivalent to showing that Λ occurs as a chain of partitions associated with a decreasing chain of subgraphs of T_Λ. This is obvious: just take the sequence of graphs $T_\Lambda \setminus \{e_1\}, T_\Lambda \setminus \{e_1, e_2\}, \ldots, T_\Lambda \setminus \{e_1, \ldots, e_{n-2}\}$. An easy induction argument shows that the partition associated with $T_\Lambda \setminus \{e_1, \ldots, e_j\}$ is λ_j, and we are done. $\qquad\square$

We want to describe a map (at least up to sign) $\mathrm{Lie}_n \to \widetilde{\mathrm{H}}^{n-3}(\Pi_n)$. As usual, we represent elements of Lie_n by parenthesized monomials in $x_1, \ldots, x_n$. There is a well-known bijection between such monomials and planar binary trees. More precisely, a planar binary tree is rooted tree, where every node has either zero or two children, called the left and right child. Nodes with zero children are called leaves, and nodes with two children are called internal. A labeling of a binary tree (by a set S) is a bijection between its set of leaves and S.

There is a bijection between the set of parenthesized multilinear monomials in a given non-empty finite set S, and the set of (isomorphism classes of) planar binary trees labeled with S. The bijection is defined as follows. If S has one element, say x, the tree associated with the monomial x is the tree with one vertex, labeled with x. Now suppose that the bijection was defined for all S with fewer than n elements. Let w be a monomial in n variables. Then necessarily $w = [u, v]$, where u and v are monomials in fewer than n variables. By hypothesis, we have associated to u and v binary trees T_u and T_v. Let T_w be the unique binary tree whose left and right subtrees are T_u and T_v respectively. This defines the bijection.

Let w be a monomial on n letters, as above, and let T_w the labeled binary tree associated with w. Let us call the labels $\{1, \ldots, n\}$. The set of internal nodes of T_w has a natural partial ordering: we say that $s < t$ if t is a descendant of s.

Definition 5.2. A *linearization* of a binary tree is a linear order on the set of its internal nodes that is compatible with the natural order.

Example 5.3. The *preorder* is the linear order of internal nodes of a binary tree that is defined recursively as follows. Assuming the tree has more than one node, list the root first, then list the internal nodes of the right subtree in preorder, then list the internal nodes of the left subtree in preorder.

To a linearization of T_w we associate a maximal chain of partitions of $\mathbf{n}$ as follows. Let $s_1, \ldots, s_{n-1}$ be the internal nodes of T_w, listed in the linear order (a binary tree with n leaves has $n - 1$ internal nodes). For $1 \leq j < n - 1$, let $T_w \setminus \{s_1, \ldots, s_j\}$ be the graph obtained from T_w by removing the nodes $s_1, \ldots, s_j$ and the edges adjacent to them. This graph is a forest that contains the leaves of T_w, which are $\{1, 2, \ldots, n\}$. Connected components $T_w \setminus \{s_1, \ldots, s_j\}$ define a partition of $\mathbf{n}$. Let us denote this partition λ_j. Clearly $\hat{0} \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} = \hat{1}$. We claim that in fact these are strict inequalities, and moreover each λ_{j+1} is obtained from λ_j by braking a single equivalence class in two

parts. Since the linear order is compatible with the natural order, s_1 must be the root of T_w. It follows that $T_w \setminus \{s_1\}$ has two connected components: the left subtree and the right subtree (we may as well assume that $n > 1$). It follows that λ_1 has two equivalence classes. For each $j \geq 1$, the parent of s_{j+1} occurs earlier than s_{j+1} in the linear order, while descendants of s_{j+1} occur later in the linear order. It follows that the connected component of s_{j+1} in $T_w \setminus \{s_1, \ldots, s_j\}$ is a binary tree with root s_j. It follows that $T_w \setminus \{s_1, \ldots, s_{j+1}\}$ is obtained from $T_w \setminus \{s_1, \ldots, s_j\}$ by braking up the connected component of s_{j+1} into the left subtree and the right subtree. It follows that λ_{j+1} is obtained from λ_j by breaking a single class in two parts. It follows that we have associated with a linearization of T_w a maximal simplex of Π_n, given by the chain $\lambda_1 < \cdots < \lambda_{n-2}$.

Remark 5.4. It is easy to show that the procedure that we just described has a kind of converse. For every maximal chain of Π_n there exists a labeled planar binary tree, together with a linearization, that induces the given maximal chain. The binary tree is unique except for the planar structure.

Lemma 5.5. *Let w be a monomial in $x_1, \ldots, x_n$, as before. The maximal chains of Π_n obtained from different choices of linearization of the binary tree T_w all project to the same element of $\mathrm{H}^{n-3}(\Pi_n)$, up to sign.*

Proof. This follows from Lemma 5.2 in the paper of Wachs [23]. Our statement is simpler than [loc. cit.], because Wachs identifies the sign precisely, while we don't need to. We remark that there are slight differences of notation and convention between our presentation and Wachs's. For example, we list chains of partitions from coarsest to finest, rather than the other way around. Because of this, the chain of partition that we get from "preorder" corresponds to the chain that Wachs gets out of a "postorder". Notwithstanding these unimportant differences, it should be clear that what we call "different linearizations of T_w" corresponds exactly to what Wachs calls "linear extensions of the postorder of the internal nodes of T". $\qquad\square$

From now on, choose a linearization of every binary planar tree labeled by $\{x_1, \ldots, x_n\}$. Suppose that w is a multilinear monomial in $x_1, \ldots, x_n$. Let T_w be the planar binary tree obtained from w. Let Λ_w be the maximal chain of Π_n that we obtained from the tree T_w using our chosen linearization. Let $[\Lambda_w] \in \mathrm{H}^{n-3}(\Pi_n)$ be the cohomology class corresponding to Λ_w. By lemma 5.5, the cohomology class $[\Lambda_w]$ is independent of the choice of linearization of T_w, up to sign. A monomial w represents an element of Lie_n, but the assignment $w \mapsto [\Lambda_w]$ does not

quite induce a homomorphism $\mathrm{Lie}_n \to \mathrm{H}^{n-3}(\Pi_n)^{\pm}$, because of the sign indeterminacy. But there is an isomorphism between these modules that is compatible with this assignment. This is the content of the following theorem.

Theorem 5.6. *For all $n \geq 1$ there is an isomorphism of Σ_n-modules*

$$\epsilon \colon \mathrm{Lie}_n \to \widetilde{\mathrm{H}}^{n-3}(\Pi_n)^{\pm}$$

At least for $n \geq 3$, the isomorphism sends every element of Lie_n represented by a monomial w either to $[\Lambda_w]$ or to $-[\Lambda_w]$.

Proof. This is a theorem of Barcelo. It follows from [23, Theorem 5.4]. □

6. The collapse map

Throughout this section, fix positive integers $n = n_1 + \cdots + n_k$. Let $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \subset \Sigma_n$ be a fixed Young subgroup. In this section we construct the $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-equivariant map that appears in the statement of the main theorem
(6.1)

$$|\Pi_n| \longrightarrow \bigvee_{d|g} \left(\bigvee_{B(\frac{n_1}{d},\ldots,\frac{n_k}{d})} \Sigma_{n_1} \times \cdots \times \Sigma_{n_k+} \wedge_{\Sigma_d} S^{n-d-1} \wedge |\Pi_d|^{\diamond} \right).$$

This map will be constructed as a collapse map associated with an embedding of a codimension zero subcomplex into Π_n. Fix a positive d that divides $n_1, \ldots, n_k$ and a monomial $w \in B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})$. In Section 4, Definition 3.4 we defined a function from $B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})$ to the set of multilinear monomials in $\frac{n}{d}$ variables. In Section 5 we showed how to associate with such a monomial a maximal chain of $\Pi_{\frac{n}{d}}$. Starting with $w \in B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})$, let $\lambda_1 < \cdots < \lambda_{\frac{n}{d}-2}$ be the associated maximal chain of $\Pi_{\frac{n}{d}}$. Let us add the indiscrete partition to this chain, to obtain the extended chain $\hat{0} = \lambda_0 < \lambda_1 < \cdots < \lambda_{\frac{n}{d}-2}$. Let us denote this extended chain Λ_w. Note that since Λ_w is a maximal chain, it is, in particular a binary chain (Definition 2.12).

Since d divides n, we may associate with it a partition with d equivalent classes, each of size $\frac{n}{d}$. Let us give a formal definition.

Definition 6.2. Suppose $0 < d|n$. Then ρ_d is the partition of $\mathbf{n}$ whose equivalence classes are

$$\left\{ 1, 1+d, 1+2d, \dots, 1 + \left(\frac{n}{d} - 1 \right) d \right\},$$
$$\left\{ 2, 2+d, 2+2d, \dots, 2 + \left(\frac{n}{d} - 1 \right) d \right\}, \dots$$
$$\dots, \{d, 2d, \dots, n\}.$$

In particular, ρ_d has d equivalence classes, each canonically isomorphic to $\frac{\mathbf{n}}{\mathbf{d}}$, via the order-preserving map.

Suppose λ is a partition of $\frac{\mathbf{n}}{\mathbf{d}}$. Define $d \times \lambda$ to be the partition of $\mathbf{n}$ that agrees with λ on each equivalence class of ρ_d, using the canonical isomorphism of the equivalence class with $\frac{\mathbf{n}}{\mathbf{d}}$. In particular, if λ_0 is the indiscrete partition of $\frac{\mathbf{n}}{\mathbf{d}}$ then $d \times \lambda_0 = \rho_d$.

Applying this operation to the chain of partitions $\hat{0} = \lambda_0 < \lambda_1 < \cdots < \lambda_{\frac{n}{d}-2}$ of $\frac{\mathbf{n}}{\mathbf{d}}$ that we defined above, we obtain the following chain of partitions of $\mathbf{n}$

$$d \times \Lambda_w = (\rho_d < d \times \lambda_1 < \cdots < d \times \lambda_{\frac{n}{d}-2}).$$

Note that $d \times \Lambda_w$ is still a binary chain of partitions. Note that in the case $d = 1$, ρ_d is the indiscrete partition, and therefore is not in Π_n.

Definition 6.3. Let Π_w be the star (in Π_n) of the chain $d \times \Lambda_w$ (for $d = 1$, Π_w is really just the chain $\lambda_1 < \cdots < \lambda_{n-2}$)

Since $d \times \Lambda_w$ is a binary chain, whose initial term is ρ_d, which is a partition with d equivalence classes, the following lemma is an immediate consequence of isomorphism (2.13) and Proposition 2.14.

Lemma 6.4. *There is a natural homeomorphism*

$$|\Pi_w| \cong \Delta^{n-d-1} * |\Pi_d|.$$

Moreover, the boundary of $|\Pi_w|$ in $|\Pi_n|$ is $S^{n-d-2} * |\Pi_d|$, where S^{n-d-2} is the topological boundary of Δ^{n-d-1}.

Moreover, the homeomorphism is Σ_d-equivariant, where Σ_d acts on $|\Pi_w|$, Δ^{n-d-1} and $|\Pi_d|$ in a natural way.

Remark 6.5. In the case $d = 1$ the lemma says that $|\Pi_w| \cong \Delta^{n-3}$. The reader may want to check this case "by hand".

The embedding $|\Pi_w| \hookrightarrow |\Pi_n|$ induces a "collapse" map

$$\Pi_n \to |\Pi_w|/\partial|\Pi_w| \xrightarrow{\cong} S^{n-d-1} * |\Pi_d|.$$

Let us digress briefly to review the generalities of this construction.

Let X be a topological space, and let $A_1, \ldots, A_m \subset X$ be closed subsets whose interiors are pairwise disjoint. Let A be the union of the A_is. There is a sequence of maps

$$(6.6) \qquad X \to X/\overline{X \setminus A} \overset{\cong}{\Leftarrow} A/\partial A \to A/\cup_i \partial A_i \overset{\cong}{\Leftarrow} \bigvee_{i=1}^m A_i/\partial A_i.$$

Here the first map is the quotient map. The second map is induced by the map of pairs $(A, \partial A) \to (X, \overline{X \setminus A})$. The third map is induced by the inclusion $\partial A \hookrightarrow \cup_i \partial A_i$. The last map is induced by the map from the disjoint union to the actual union inside X

$$\coprod_i A_i \to \bigcup_i A_i.$$

It is easy to see that under the assumption that the A_is only intersect at boundary points, the last map is a homeomorphism. Since the "wrong way" maps in (6.6) are homeomorphisms, we obtain a well defined map $X \to \bigvee_{i=1}^m A_i/\partial A_i$. We refer to it as the collapse map associated with the inclusions $A_i \hookrightarrow X$.

To go back to the relevant example, the embedding $|\Pi_w| \hookrightarrow |\Pi_n|$ induces a Σ_d-equivariant quotient map

$$q_w \colon \Pi_n \to |\Pi_w|/\partial|\Pi_w| \overset{\cong}{\to}$$
$$\overset{\cong}{\to} (\Delta^{n-d-1} * |\Pi_d|)/(S^{n-d-2} * |\Pi_d|) \overset{\cong}{\to} S^{n-d-1} * |\Pi_d|.$$

Our next task is to compare the induced map in cohomology with the homomorphism $\phi_w \colon \mathrm{Lie}_d \to \mathrm{Lie}_n$ that we constructed for each $w \in B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})$ (ϕ_w is a specialization of (4.4)). Recall that there is an isomorphism (for each n) of Σ_n-modules $\epsilon_n \colon \mathrm{Lie}_n \to \mathrm{H}^{n-3}(\Pi_n)^{\pm}$ (Theorem 5.6).

There are two natural suspension isomorphisms $\mathrm{H}^{d-3}(|\Pi_d|) \to \mathrm{H}^{n-3}(S^{n-d-1} * |\Pi_d|)$. Choose one and call it σ.

Consider the following (not necessarily commutative) diagram

$$
\begin{array}{ccc}
\mathrm{Lie}_d & \xrightarrow{\quad\quad\quad \phi_w \quad\quad\quad} & \mathrm{Lie}_n \\
\downarrow{\scriptstyle \epsilon_d} & & \downarrow{\scriptstyle \epsilon_n} \\
\widetilde{\mathrm{H}}^{d-3}(|\Pi_d|) \xrightarrow{\ \sigma\ } \widetilde{\mathrm{H}}^{n-3}(S^{n-d-1} * |\Pi_d|) & \xrightarrow{\ q_w^* \ } & \widetilde{\mathrm{H}}^{n-3}(|\Pi_n|)
\end{array}
$$

Lemma 6.7. *Let w_d be a parenthesized multilinear monomial in $x_1, \ldots, x_d$. It represents an element of Lie_d. The images of this element in $\widetilde{\mathrm{H}}^{n-3}(|\Pi_n|)$ under the two maps around the diagram are the same up to sign.*

Proof. In what follows, suppose that $d \geq 3$. The argument needs some minor adjustments for $d = 1, 2$. We will do these cases at the end of the proof.

Consider first the image of w_d in Lie_n under ϕ_w. By (4.5), it is represented by the monomial

$$w_d(\tilde{w}(x_1, x_{1+d}, ..., x_{1+n-d}), \tilde{w}(x_2, x_{2+d}, ..., x_{2+n-d}), \ldots, \tilde{w}(x_d, x_{2d}, ..., x_n)),$$

where $\tilde{w}$ is the resolution of w (Definition 3.4). The image of this monomial in $\mathrm{H}^{n-3}(|\Pi_n|)$ is represented up to sign (throughout the proof, "represented" means "represented up to sign") by a maximal simplex associated with the planar binary tree associated with this monomial. Given a monomial w, let T_w be the associated binary tree. It is easy to see that the binary tree for our monomial is obtained by grafting a copy of $T_{\tilde{w}}$ onto each leaf of T_{w_d}, and then labeling the leaves of the i-th copy of $T_{\tilde{w}}$ by $i, i + d, i + 2d, \ldots$. Let us call this tree $T_{w_d \circ \tilde{w}}$.

Going the other way around the diagram, the image of w_d in $\widetilde{\mathrm{H}}^{d-3}(|\Pi_d|)$ is represented by a maximal simplex associated with the planar binary tree T_{w_d}. Let's denote this maximal simplex Λ_{w_d}. The image of $[\Lambda_{w_d}]$ in $\widetilde{\mathrm{H}}^{n-3}(S^{n-d-1} * |\Pi_d|)$ is represented by a maximal simplex that is the join of Λ_{w_d} with a maximal simplex of S^{n-d-1}. The image in $\widetilde{\mathrm{H}}^{n-3}(|\Pi_n|)$ is then represented by a maximal simplex that is the join of Λ_{w_d}, which is now considered a maximal simplex of $(\Pi_w)_{<\rho_s} \cong \Pi_d$ and a maximal simplex of $(\Pi_w)_{\geq \rho_d}$. It is easy to see that any such join of maximal simplices is obtained from some linearization of the tree $T_{w_d \circ \tilde{w}}$. A linearization in which the interval vertices of T_{w_d} precede the internal vertices of the copies of $\tilde{w}$. It follows that the two images of w_d in Lie_n are represented by two linearizations of the same planar binary tree $T_{w_d \circ \tilde{w}}$. It follows by Lemma 5.5 that they are the same up to sign.

In the case $d = 1$ or 2, observe that the image of a generator of Lie_d in $\widetilde{\mathrm{H}}^{n-3}(S^{n-d-1} * |\Pi_d|)$ is still represented by a maximal simplex of $\mathrm{St}(\Lambda_w)$. The rest of the proof remains the same as before. $\square$

Now let's go back to the situation where $n = n_1 + \cdots + n_k$. For every positive d that divides $n_1, \ldots, n_k$ we have a Σ_d-equivariant homomorphism, induced by the collapse maps defined above.

$$\bigoplus_{B(\frac{n_1}{d}, ..., \frac{n_k}{d})} \mathrm{H}^{n-3}(S^{n-d-1} * |\Pi_d|) \to \mathrm{H}^{n-3}(|\Pi_n|).$$

This homomorphism can be induced up to a $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-equivariant homomorphism. Taking direct sum over all d that divide $n_1, \ldots, n_k$, one obtains the homomorphism referred to in the following proposition.

Proposition 6.8. *The following homomorphism, induced by the collapse maps that we defined, is an isomorphism*

$$\bigoplus_{\substack{d \mid \gcd(n_1,\ldots,n_k) \\ w \in B(\frac{n_1}{d},\ldots,\frac{n_k}{d})}} \mathrm{H}^{n-3}(S^{n-d-1} * |\Pi_d|) \otimes_{\Sigma_d} \mathbb{Z}[\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}] \to \mathrm{H}^{n-3}(|\Pi_n|).$$

Proof. For typographical reasons, we introduce the following notation. Suppose M is a (right) $\mathbb{Z}[\Sigma_d]$-module. Then

$$M{\uparrow} := M \otimes_{\mathbb{Z}[\Sigma_d]} \mathbb{Z}[\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}].$$

Consider the following diagram of $\mathbb{Z}[\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}]$-modules. It is closely related to the diagram preceding Lemma 6.7. For typographical reasons, we omit the indexing sets under the $\bigoplus$ signs. All the direct sums are taken over all positive $d \mid \gcd(n_1,\ldots,n_k)$, and for each such d, over all $w \in B(\frac{n_1}{d},\ldots,\frac{n_k}{d})$

$$
\begin{array}{ccc}
\bigoplus \mathrm{Lie}_d{\uparrow} & \xrightarrow{\quad\Sigma\phi_w\quad} & \mathrm{Lie}_n \\
\downarrow{\scriptstyle \oplus\epsilon_d} & & \downarrow{\scriptstyle \epsilon_n} \\
\bigoplus \mathrm{H}^{d-3}(|\Pi_d|){\uparrow} \xrightarrow{\ \sigma\ } \bigoplus \mathrm{H}^{n-3}(S^{n-d-1} * |\Pi_d|){\uparrow} \xrightarrow{\Sigma q_w^*} & & \mathrm{H}^{n-3}(|\Pi_n|)
\end{array}
$$

Let $w \in B(\frac{n_1}{d},\ldots,\frac{n_k}{d})$, θ_w a monomial representing an element of Lie_d, and $\sigma \in \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$. Then $[\theta_w] \otimes \sigma$ represents an element of the top left corner, and elements of this form provide a generating set for the top left corner. The top homomorphism in the diagram is an isomorphism by Theorem 3.9 (the Hilton-Milnor theorem). The vertical homomorphisms are induced by the standard isomorphism from Lie_n to the cohomology of the partition poset (Theorem 5.6). It follows that a generating set for the top left corner will be mapped to a generating set of the bottom right corner via the top and right maps. It follows from Lemma 6.7 that elements of the form $[\theta_w \otimes \sigma]$ are mapped to the same element of $\mathrm{H}^{n-3}(\Pi_n)$ by the top and right maps as by the left and bottom maps, except possibly for some negative signs. Therefore, the set of elements of the form $[\theta_w \otimes \sigma]$ is mapped to a generating set of the bottom right corner via the left and bottom maps. It follows that the composed map on the bottom is an epimorphism. Since it is a homomorphism between free abelian groups of the same rank, it is an isomorphism. Since the left bottom map is clearly an isomorphism, it follows that the right bottom map is an isomorphism as well. This is what we wanted to prove. $\qquad\square$

Suppose that $d|\gcd(n_1,\ldots,n_k)$, and $w \in B(\frac{n_1}{d},\ldots,\frac{n_k}{d})$. Then Π_w is a Σ_d-invariant subposet of Π_n. Let $\sigma \in \Sigma_d \backslash \Sigma_{n_1} \times \ldots \times \Sigma_{n_k}$ be a coset. Then $\Pi_w \cdot \sigma$ is a well-defined subposet of Π_n.

Lemma 6.9. *Suppose that $d, d'|\gcd(n_1,\ldots,n_k)$, $w \in B(\frac{n_1}{d},\ldots,\frac{n_k}{d})$, $w' \in B(\frac{n_1}{d'},\ldots,\frac{n_k}{d'})$, $\sigma \in \Sigma_d \backslash \Sigma_{n_1} \times \ldots \times \Sigma_{n_k}$, $\sigma' \in \Sigma_{d'} \backslash \Sigma_{n_1} \times \ldots \times \Sigma_{n_k}$. Suppose, furthermore, that $(d, w, \sigma) \neq (d', w', \sigma')$. Then the subcomplexes $\Pi_w \cdot \sigma$ and $\Pi_{w'} \cdot \sigma'$ of Π_n have disjoint sets of maximal simplices.*

Proof. Suppose that Δ and Δ' are maximal simplices of Π_w and $\Pi_{w'}$ respectively. It follows from Lemma 5.1 that Δ and Δ' represent non-zero classes in the groups $\mathrm{H}^{n-3}(S^{n-d-1} * |\Pi_d|)$ and $\mathrm{H}^{n-3}(S^{n-d'-1} * |\Pi_{d'}|)$ corresponding to w and w'. By Proposition 6.8, the following homomorphism is an isomorphism

$$\bigoplus \mathrm{H}^{n-3}(S^{n-d-1} * |\Pi_d|) \otimes_{\Sigma_d} \mathbb{Z}[\Sigma_{n_1} \times \ldots \times \Sigma_{n_k}] \to \mathrm{H}^{n-3}(\Pi_n).$$

Since $(d, w, \sigma) \neq (d', w', \sigma')$, $[\Delta] \otimes \sigma$ and $[\Delta'] \otimes \sigma'$ represent non-zero elements in two different summands of the source. It follows that $\Delta \cdot \sigma$ and $\Delta' \cdot \sigma'$ represent different elements of $\mathrm{H}^{n-3}(\Pi_n)$. Therefore, they can not be the same maximal simplex. $\square$

Proposition 6.10. *The inclusions $\Pi_w \hookrightarrow \Pi_n$ induce a quotient map*

$$(6.11) \quad |\Pi_n| \longrightarrow \bigvee_{d|g} \left(\bigvee_{B(\frac{n_1}{d},\ldots,\frac{n_k}{d})} \Sigma_{n_1} \times \cdots \times \Sigma_{n_{k+}} \wedge_{\Sigma_d} S^{n-d-1} * |\Pi_d| \right).$$

This map is $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-equivariant, and it induces an isomorphism in cohomology.

Remark 6.12. By Lemma 2.5, $S^{n-d-1} * |\Pi_d|$ is naturally equivalent to $S^{n-d-1} \wedge |\Pi_d|^\diamond$. Therefore, the map (6.11) determines a map as in (6.1).

Proof. Consider all the subcomplexes $\Pi_w \cdot \sigma \subset \Pi_n$, as w ranges over all elements of $B(\frac{n_1}{d},\ldots,\frac{n_k}{d})$ (for all d), and σ ranges over all cosets $\Sigma_d \backslash \Sigma_{n_1} \times \ldots \times \Sigma_{n_k}$. By Lemma 6.9, these subcomplexes do not share any maximal simplices. It follows that they only may intersect at boundary points. Therefore we obtain a collapse map, as in (6.6)

$$|\Pi_n| \to \bigvee_{d|g} \bigvee_{B(\frac{n_1}{d},\ldots,\frac{n_k}{d})} \Sigma_{n_1} \times \cdots \times \Sigma_{n_{k+}} \wedge_{\Sigma_d} |\Pi_w|/\partial|\Pi_w|.$$

It follows from Lemmas 6.4 and 2.11 that $|\Pi_w|/\partial|\Pi_w| \cong S^{n-d-1} * |\Pi_d|$, and we obtain the desired map. The induced homomorphism in cohomology is the one of Proposition 6.8, and so it is an isomorphism. $\square$

Example 6.13. Let us describe explicitly the map (6.1) or equivalently the map (6.11) for the case of the group $\Sigma_2 \times \Sigma_2$ acting on $|\Pi_4|$. The possible values of d are 1 and 2. The set $B(2,2)$ consists of one element: the monomial $w_2 := [[[x_2, x_1], x_1], x_2]$. The resolution of w_2 is the monomial $[[[x_3, x_1], x_2], x_4]$. The associated chain of partitions is $123/4 < 13/2/4$ (in this case the binary tree associated with the monomial has only one linearization, so there is only one possible maximal simplex to associate with w_2). Since this is a maximal chain, the star of the chain is the chain itself. So Π_{w_2} is the chain $123/4 < 13/2/4$. The poset $\Sigma_2 \times \Sigma_2 \times \Pi_{w_2}$ is the union of the following four chains:

$$(123/4 < 13/2/4), (123/4 < 23/1/4), (124/3 < 14/2/3), (124/3 < 24/1/3)$$

The set $B(1,1)$ also consists of one element: the monomial $w_1 := [x_2, x_1]$. The associated partition ρ_2 is $13/24$. The poset Π_{w_1} consists of all partitions comparable with this one. It can be described as follows

$$1/3/24 > 13/24 < 13/2/4.$$

As expected, Π_{w_1} is a Σ_2-invariant subposet of Π_4. The poset $\Sigma_2 \times \Sigma_2 \times_{\Sigma_2} \Pi_{w_1}$ is the union of the poset above with the poset

$$2/3/14 > 23/14 < 23/1/4.$$

The reader is invited to check that the geometric realization of the subcomplex

$$|\Sigma_2 \times \Sigma_2 \times \Pi_{w_2} \coprod \Sigma_2 \times \Sigma_2 \times_{\Sigma_2} \Pi_{w_1}| \subset |\Pi_4|$$

is exactly the subcomplex marked with fat lines on Figure 1.4. It follows that the map (6.11) is, in this case, the same one as defined in that figure. Since the map is defined by collapsing a contractible subcomplex, it is in fact a homotopy equivalence (and not just a cohomology equivalence).

7. Fixed points

Let G be a group acting (on the left) on the set $\mathbf{n}$. Specifying such an action is equivalent to specifying a homomorphism $G \to \Sigma_n$. Sometimes we will need to assume that the action is effective, which means that G injects into Σ_n. The action of G on $\mathbf{n}$ induces an action on Π_n. Our goal in this section is to describe the fixed points space $|\Pi_n|^G \cong |\Pi_n^G|$.

The orbits of the action of G partition the set $\mathbf{n}$ into G-invariant subsets. Each orbit is isomorphic to a set of the form G/H, where H is a subgroup of G, determined up to conjugation. We say that the action of G on $\mathbf{n}$ is isotypical, if all the G-orbits are isomorphic as G-sets. We

suspect that the following simple lemma was observed by others, but we do not know a reference.

Lemma 7.1. *If the action of G on $\mathbf{n}$ is not isotypical, then $|\Pi_n|^G$ is contractible.*

Proof. Let θ be the canonical partition of $\mathbf{n}$ into isotypical components. Specifically, x and y belong to the same equivalence class of θ if the G-orbits of x and y are isomorphic G-sets. Since G is not isotypical, θ has at least 2 equivalence classes. It is also clear that θ has fewer than n classes, because otherwise all the classes would be singletons, which would mean that G acts trivially, and in particular isotypically.

Let $\lambda \in \Pi_n^G$ be a non-trivial G-invariant partition. Let $\theta \vee \lambda$ be the coarsest common refinement of θ and λ. It is the partition whose equivalence classes are all the intersections of classes of θ with classes of λ. We claim that $\theta \vee \lambda$ is not the discrete partition. Indeed, suppose that $\theta \vee \lambda$ is the discrete partition. Let c be a non-singleton equivalence class of λ. If $\theta \vee \lambda$ is the discrete partition then different elements of c belong to different classes of θ. Let x, y be two elements of c. Since they belong to different classes of θ, the G-orbits Gx and Gy are not isomorphic. It follows that the stabilizer groups of x and y are not the same. In particular, there exists a $g \in G$ that stabilizes one but not the other. Suppose that $gx = x$ but $gy \neq y$. Since λ is G-invariant, it follows that $gx \sim gy$ in λ. But $gx = x$ and $x \sim y$, so $y \sim gy$ in λ. But y and gy are in the same class of θ, so they are in the same class of $\lambda \vee \theta$, contradicting that $\lambda \cap \theta$ is discrete.

It follows that $\theta \vee \lambda$ is an element of Π_n^G, for all $\lambda \in \Pi_n^G$. By lemma 2.3, the space $|\Pi_n^G|$ is contractible. $\square$

It remains to analyze the fixed points of isotypical actions. An important special case is that of transitive actions. Suppose G acts transitively on $\mathbf{n}$. Then $\mathbf{n}$ can be identified (non-canonically) with the set G/H for some subgroup $H \subset G$. So it is enough to describe $(\Pi_{G/H})^G$. The following is essentially [25, Lemma 3].

Lemma 7.2. *The poset of G-invariant partitions of G/H is isomorphic to the poset of subgroups $H \subseteq K \subseteq G$, ordered by reverse inclusion. The non-trivial partitions correspond to subgroups $H \subsetneq K \subsetneq G$.*

Proof. Let λ be a G-invariant partition of G/H. Let $K = \{g \in G \mid g(eH) \sim_\lambda eH\}$. Then K is a subgroup of G. Indeed, suppose that $eH \sim_\lambda g_1 H \sim_\lambda g_2 H$. Multiplying the first equivalence by g_2 we obtain that $g_2 H \sim_\lambda g_2 g_1 H$. Since $g_2 H \sim_\lambda eH$, it follows that $g_2 g_1 \in K$. This shows that K is closed under multiplication. A similar argument shows

that K is closed under taking inverses. Clearly, $H \subseteq K$. It is easy to see that a finer partition will lead to a smaller group K.

Conversely, let K be a group, $H \subseteq K \subseteq G$. Let λ be the partition of G/H defined by the rule $g_1 H \sim_\lambda g_2 H$ if and only if $g_2^{-1} g_1 \in K$. It is easy to check that λ is a well defined G-invariant partition, and that we defined two maps between posets that are inverse to each other. Finally, it is clear that $K = H, G$ correspond to the discrete and the indiscrete partition respectively. □

Example 7.3. Consider the standard action of Σ_n on $\mathbf{n}$. This can be identified with the action of Σ_n on Σ_n / Σ_{n-1}. Since there are no groups $\Sigma_{n-1} \subsetneq H \subsetneq \Sigma_n$, it follows that $\Pi_n^{\Sigma_n} = \emptyset$.

Example 7.4. Suppose that $n = k_1 \cdot k_2 \cdots k_l$, with $l > 1$ and all factors greater than 1. Consider the iterated wreath product $W :=$ $\Sigma_{k_1} \wr \cdots \wr \Sigma_{k_l} \subset \Sigma_n$. This group fits into a normal series

$$\{e\} \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_l = W$$

where $G_i / G_{i-1} = (\Sigma_{k_i})^{k_{i+1} \cdots k_l}$. W acts transitively on $\mathbf{n}$, with stabilizer groups (conjugate to) $W \cap \Sigma_{n-1}$.

Lemma 7.5. *Let W be an iterated wreath product as above. The space $|\Pi_n|^W$ is contractible.*

Proof. By Lemma 7.2, the poset $(\Pi_n)^W$ is (reverse) isomorphic to the poset of groups G that satisfy $W \cap \Sigma_{n-1} \subsetneq G \subsetneq W$. So it is enough to show that this poset of groups has contractible nerve. We sill show that it has a minimal element.

Let H be the subgroup of W generated by $W \cap \Sigma_{n-1}$ and the transposition $(n-1, n)$. Explicitly,

$$H = \Sigma_{k_1} \wr \cdots \wr \Sigma_{k_l - 1} \times \Sigma_{k_1} \wr \cdots \wr \Sigma_{k_{l-1} - 1} \times \cdots \Sigma_{k_1} \wr \Sigma_{k_2 - 1} \times \Sigma_{k_1}.$$

We claim that every group G satisfying $W \cap \Sigma_{n-1} \subsetneq G \subset W$ contains H. Indeed, let G be such a group. We need to show that G contains the transposition $(n-1, n)$. By assumption, G contains an element $\sigma \in W \setminus \Sigma_{n-1}$. Thus $\sigma(n) \neq n$. Suppose first that $\sigma(n) \in \{n - k_1 + 1, \ldots, n\}$. Then, since $\sigma \in W$, σ leaves invariant the set $\{n - k_1 + 1, \ldots, n\}$. Let $\overline{\sigma}$ be the permutation of $\mathbf{n}$ that agrees with σ on the set $\{n - k_1 + 1, \ldots, n\}$, and is the identity on the complement of this set. Clearly $\overline{\sigma} \in W$ and $\sigma^{-1} \overline{\sigma} \in W \cap \Sigma_{n-1} \subset G$. It follows that $\overline{\sigma} \in G$. Thus G contains all the permutations of the set $\{n - k_1 + 1, \ldots, n - 1\}$, plus an additional permutation of the set $\{n - k_1 + 1, \ldots, n\}$. Therefore, G contains the full group of permutations of $\{n - k_1 + 1, \ldots, n\}$, which is what we wanted to show.

Now suppose that $\sigma(n) \in \{1, \ldots, n - k_1\}$. Then $\sigma(n) \in \{jk_1 + 1, jk_1 + 2, \cdots, (j+1)k_1\}$, where $jk_1 < n - k_1$. Then $\sigma(n-1) \in \{jk_1 + 1, jk_1 + 2, \cdots, (j+1)k_1\}$ as well. Let τ be the transposition $(\sigma(n), \sigma(n-1))$. Clearly, $\tau \in W \cap \Sigma_{n-1} \subset G$. Thus $\sigma^{-1}\tau\sigma \in G$. But $\sigma^{-1}\tau\sigma$ is the transposition $(n-1, n)$. $\square$

It remains to analyze the fixed points of a general isotypical action. Our strategy is to first analyze the fixed points of groups that occur as isotropy groups of the spaces in (1.3). This will enable us to prove the main theorem. That, in turn, will enable us to describe the fixed points of Π_n for all subgroups of Σ_n.

First, some easy generalities. Suppose G acts on $\mathbf{n}$. Let λ be a partition of $\mathbf{n}$ preserved by G. Let S and T be two G-orbits in $\mathbf{n}$. We say that S and T are connected by λ if there exist $s \in S$ and $t \in T$ such that $s \sim_\lambda t$. The proof of the following elementary lemma is left to the reader.

Lemma 7.6. *The relation of being connected by λ is an equivalence relation on the set of G-orbits.*

Lemma 7.7. *Suppose G acts isotypically on $\mathbf{n}$, λ is a G-invariant partition, and S and T are orbits of G connected by λ then the restrictions of λ to S and T are isomorphic.*

Proof. Let $s \in S$ and $t \in T$ be such that $s \sim_\lambda t$. Let $K_s = \{g \in G \mid gs \sim_\lambda s\}$ and $K_t = \{g \in G \mid gt \sim_\lambda t\}$. Since λ is G-invariant, and $s \sim_\lambda t$, it follows that for all $g \in G$, $gs \sim_\lambda gt$. Therefore, $gs \sim s$ if and only if $gt \sim t$. In other words, $K_s = K_t$. By the proof of Lemma 7.2, K_s and K_t determine the restrictions of λ to the transitive G-sets S and T. $\square$

Let us suppose that $d|n$ and d factors as a product of positive integers $d = d_1 \cdots d_l$ with $l > 1$. Consider the iterated wreath product $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$ as a subgroup of Σ_n via the inclusions

$$\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l} \subset \Sigma_d \subset (\Sigma_d)^{\frac{n}{d}} \subset \Sigma_n.$$

Lemma 7.8. *The fixed points space $|\Pi_n|^{\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}}$ is contractible.*

Proof. The orbits of $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$ are the sets

$$\{1, \ldots, d\}, \{d+1, \ldots, 2d\}, \ldots, \{n-d+1, \ldots, n\}.$$

Let θ be the partition of $\mathbf{n}$ whose equivalence classes are $\{1, \ldots, d_1\}, \{d_1 + 1, \ldots, 2d_1\}, \ldots, \{n - d_1 + 1, \ldots, n\}$. Then θ is a $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$-invariant partition. It follows from the proof of Lemma 7.5 that the restriction of θ to each orbit of $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$ is the finest non-discrete $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$-invariant partition of the orbit. Let λ be another $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$-invariant

partition of $\mathbf{n}$. Let $\lambda \wedge \theta$ be the finest common coarsening of λ and θ. Clearly, $\lambda \wedge \theta$ is a $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$-invariant partition. We claim that if λ is not indiscrete then $\lambda \wedge \theta$ is not indiscrete. To see this, suppose first that there exists two orbits S and T of $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$ that are not connected by λ. Since each equivalence class of θ is contained in some orbit of $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$, there do not exists elements $s \in S$ and $t \in T$ that are in same equivalence class of $\lambda \wedge \theta$. In particular, $\lambda \wedge \theta$ is not indiscrete.

Now suppose that all orbits of $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$ are connected by λ. Then by Lemma 7.7 the restrictions of λ to the orbits of $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$ are pairwise isomorphic. If the restriction of λ to each orbit is indiscrete, then (since we assume that all the orbits are connected) λ is indiscrete. Conversely, if λ is not indiscrete, then the restrictions of λ to the orbits of $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$ are not indiscrete. Since the restriction of θ to an orbit of $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$ is the finest non-discrete invariant partition of the orbit, it follows that the restriction of λ to each orbit of $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$ is either discrete, or is coarser than the restriction of θ to the orbit. In the first case, the restriction of $\lambda \wedge \theta$ to each orbit of $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$ is the same as the restriction of θ to the orbit. In the second case, it is the same as the restriction of λ to the orbit. In any case, the restriction of $\lambda \wedge \theta$ to each orbit of $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$ is not indiscrete. It follows that $\lambda \wedge \theta$ is not indiscrete.

It follows from the claim that $\lambda \wedge \theta$ is an element of $\Pi_n^{\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}}$ for all $\lambda \in \Pi_n^{\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}}$. By lemma 2.3 it follows that the geometric realization of $\Pi_n^{\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}}$ is contractible. $\qquad\square$

Suppose G acts isotypically on $\mathbf{n}$, so all G-orbits are isomorphic. Let us say that each orbit has d elements, so there are $\frac{n}{d}$ orbits. Let us fix a transitive action of G on $\mathbf{d}$, isomorphic to the action of G on one of the orbits. In particular, we have fixed a homomorphism $G \to \Sigma_d$. Let us also fix a Σ_d-equivariant isomorphism $\mathbf{d} \times \frac{\mathbf{n}}{\mathbf{d}} \overset{\cong}{\to} \mathbf{n}$. Without loss of generality we can take this isomorphism to be $(i, j) \mapsto i + (j-1)d$. Recall that Part_n is the full poset of partitions of $\mathbf{n}$. Consider the following composition of Σ_d-equivariant maps of posets

$$\mathrm{Part}_d \times \mathrm{Part}_{\frac{n}{d}} \to \mathrm{Part}_{\mathbf{d} \times \frac{\mathbf{n}}{\mathbf{d}}} \to \mathrm{Part}_n \,.$$

Here the first map takes cartesian product of equivalence relations, and the second map uses the bijection $\mathbf{d} \times \frac{\mathbf{n}}{\mathbf{d}} \overset{\cong}{\to} \mathbf{n}$. Removing the initial and final elements, we obtain a map of posets

$$\overline{\mathrm{Part}_d \times \mathrm{Part}_{\frac{n}{d}}} \to \Pi_n.$$

By Proposition 2.8, there are homeomorphisms

$$|\overline{\mathrm{Part}_d \times \mathrm{Part}_{\frac{n}{d}}}| \cong \Sigma|\Pi_d| * |\Pi_{\frac{n}{d}}|$$

Thus we obtain a Σ_d-equivariant map of spaces

$$\Sigma|\Pi_d| * |\Pi_{\frac{n}{d}}| \to |\Pi_n|.$$

Note that the partition of $\mathbf{n}$ into the orbits of Σ_d determines a Σ_d-invariant point of $|\overline{\mathcal{P}_d \times \mathcal{P}_{\frac{n}{d}}}|$, and a $(\Sigma_d)^{\frac{n}{d}}$-invariant point of $|\Pi_n|$. The map above preserves these basepoints. Therefore it induces a $(\Sigma_d)^{\frac{n}{d}}$-equivariant map

$$\Sigma_d \times \cdots \times \Sigma_{d+} \wedge_{\Sigma_d} \Sigma|\Pi_d * \Pi_{\frac{n}{d}}| \to |\Pi_n|.$$

Viewing G as a subgroup of $(\Sigma_d)^{\frac{n}{d}}$, we obtain a map of G-fixed points

$$(7.9) \qquad (\Sigma_d \times \cdots \times \Sigma_{d+} \wedge_{\Sigma_d} \Sigma|\Pi_d * \Pi_{\frac{n}{d}}|)^G \to |\Pi_n|^G.$$

Recall that $C_{\Sigma_d}(G)$ is the centralizer of G in Σ_d. Clearly, the centralizer of G in $(\Sigma_d)^{\frac{n}{d}}$ is $C_{\Sigma_d}(G)^{\frac{n}{d}}$. Note also that $\Sigma|\Pi_d * \Pi_{\frac{n}{d}}|^G \cong \Sigma|\Pi_d|^G * \Pi_{\frac{n}{d}}|$. By corollary 2.19, there is a homeomorphism

$$(7.10) \quad C_{\Sigma_d}(G)^{\frac{n}{d}}_+ \wedge_{C_{\Sigma_d}(G)} \Sigma|\Pi_d|^G * |\Pi_{\frac{n}{d}}| \xrightarrow{\cong} \left((\Sigma_d)^{\frac{n}{d}}_+ \wedge_{\Sigma_d} \Sigma|\Pi_d * \Pi_{\frac{n}{d}}|\right)^G.$$

Combine (7.10) with (7.9) to obtain the following map:

$$(7.11) \qquad C_{\Sigma_d}(G)^{\frac{n}{d}}_+ \wedge_{C_{\Sigma_d}(G)} \Sigma|\Pi_d|^G * |\Pi_{\frac{n}{d}}| \to |\Pi_n|^G.$$

Lemma 7.12. *Suppose $G = \Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$, where $d = d_1 \cdots d_l$, $l \geq 1$ (note that we include $l = 1$). The map (7.11) is a homotopy equivalence.*

Remark 7.13. We will need Lemma 7.12 to prove our main result. Later we will use the main result to prove that the conclusion of the lemma holds when G is any transitive subgroup of Σ_d.

Lemma 7.14. *Suppose, as usual, that $d|n$ and G is a transitive subgroup of Σ_d, embedded diagonally in Σ_n. Let T be the set of G-invariant partitions of $\mathbf{n}$ with the property that each equivalence class intersects each G-orbit at a single element. There are bijections*

$$T \cong C_{\Sigma_d}(G)^{\frac{n}{d}-1} \cong C_{\Sigma_d}(G)^{\frac{n}{d}}/C_{\Sigma_d}(G).$$

Proof. By our assumptions the action of G on $\mathbf{n}$ has $\frac{n}{d}$ orbits, each isomorphic to $\mathbf{d}$. Let $O_1, \ldots, O_{\frac{n}{d}}$ be the orbits of G. Let $\lambda \in T$ be a partition of $\mathbf{n}$ that intersects each orbit at a single element. Then for each $i = 2, \ldots, \frac{n}{d}$, λ determines a G-equivariant bijection $O_1 \to O_i$ which associates to each element of O_1 the unique element of O_i that is in the same equivalence class of λ. The set of G-equivariant bijections

from O_1 to O_i is canonically isomorphic to $C_{\Sigma_d}(G)$. It follows that λ determines an element of $C_{\Sigma_d}(G)^{\frac{n}{d}-1}$. We have defined a map $T \to C_{\Sigma_d}(G)^{\frac{n}{d}-1}$. It is easy to check that this map is a bijection. $\square$

Proof of Lemma 7.12. Consider first the case $l > 1$, so that we have a wreath product of at least two factors. In that case $|\Pi_d|^G$ is contractible by Lemma 7.5, and $|\Pi_n|^G$ is contractible by Lemma 7.8. It follows that the source and the target of the map (7.11) are contractible, and so the map is a homotopy equivalence.

It remains to analyze the case $l = 1$, or in other words, $G = \Sigma_d$. Our goal is to prove that the map (7.11) is an equivalence. We will now analyze the homotopy type of the target of the map, i.e, of the space $|\Pi_n|^{\Sigma_d}$.

Before we proceed, it is worth noting that $C_{\Sigma_d}(\Sigma_d)$ is the center of Σ_d. We sometimes may write it simply as $C(\Sigma_d)$. This group is trivial if $d \neq 2$ and is Σ_2 when $d = 2$. However, we will not separate the cases, but just write $C(\Sigma_d)$ as per the general case. Note also that $C(\Sigma_d)$ acts trivially on $|\Pi_d|^{\Sigma_d} * |\Pi_{\frac{n}{d}}|$. Therefore, there is a natural isomorphism

$$(7.15) \qquad C(\Sigma_d)^{\frac{n}{d}}_+ \wedge_{C(\Sigma_d)} \Sigma|\Pi_d|^{\Sigma_d} * |\Pi_{\frac{n}{d}}| \cong T_+ \wedge \Sigma|\Pi_d|^{\Sigma_d} * |\Pi_{\frac{n}{d}}|.$$

Recall that $T \subset (\Pi_n)^{\Sigma_d}$ is the set of Σ_d-invariant partitions that intersect each Σ_d orbit at a single element (in particular, every $\lambda \in T$ has d equivalence classes, with $\frac{n}{d}$ elements in each). Let $\Pi^{\Sigma_d}_{\leq T \leq}$ be the poset of Σ_d-invariant partitions of **n** that are either finer or coarser than at least one element of T.

Let $\lambda_0 < \cdots < \lambda_n$ be a chain of Σ_d-invariant partitions. We claim that either the entire chain is contained in the poset $\Pi^{\Sigma_d}_n \setminus T$ or it is contained in $\Pi^{\Sigma_d}_{\leq T \leq}$. Indeed, suppose the entire chain is not contained in $(\Pi_n)^{\Sigma_d} \setminus T$. Then $\lambda_i \in T$ for some λ_i in the chain. But then it is clear that the entire chain is in $\Pi^{\Sigma_d}_{\leq T \leq}$.

It follows from this claim that $|\Pi_n|^{\Sigma_d} \cong |\Pi^{\Sigma_d}_n \setminus T| \cup |\Pi^{\Sigma_d}_{\leq T \leq}|$, and moreover that $|\Pi_n|^{\Sigma_d}$ is equivalent to the homotopy pushout of the diagram

$$(7.16) \qquad |\Pi^{\Sigma_d}_n \setminus T| \leftarrow |(\Pi^{\Sigma_d}_n \setminus T) \cap \Pi^{\Sigma_d}_{\leq T \leq}| \rightarrow |\Pi^{\Sigma_d}_{\leq T \leq}|.$$

Next we analyze $|\Pi^{\Sigma_d}_n \setminus T|$. Specfically, we will show that it is contractible. Let $\lambda \in \Pi^{\Sigma_d}_n \setminus T$. Then in particular λ is a Σ_d-invariant partition of **n**. The restriction of λ to an orbit of Σ_d defines a Σ_d-invariant partition of this orbit. Since each orbit is isomorphic to **d** with the standard action of Σ_d, the restriction of λ to each orbit is either the discrete or the indiscrete partition of that orbit (this follows from Example 7.3). We claim that not all the orbits of Σ_d are connected

by λ. Indeed, suppose that all orbits are connected. Then the restriction of λ to each orbit is either indiscrete for all orbits, or discrete for all orbits. In the first case, λ is itself indiscrete, in which case it is not in Π_n. In the second case, $\lambda \in T$, contradicting our assumption. Let θ be the partition of $\mathbf{n}$ into Σ_d-orbits, and let $\lambda \wedge \theta$ be the finest common coarsening of λ and θ. Since not all the classes of θ are connected by λ, it follows that $\lambda \wedge \theta$ is not the indiscrete partition. By Lemma 2.3 $|\Pi_n^{\Sigma_d} \setminus T|$ is a contractible space.

Since $|\Pi_n^{\Sigma_d} \setminus T|$ is contractible, the space $|\Pi_n|^{\Sigma_d}$, which is equivalent to the homotopy pushout on line (7.16), is homotopy equivalent to the cofiber (as well as homotopy cofiber) of the inclusion map

$$|(\Pi_n^{\Sigma_d} \setminus T) \cap \Pi_{\leq T \leq}^{\Sigma_d}| \to |\Pi_{\leq T \leq}^{\Sigma_d}|.$$

For an element $t \in T$. Let $\Pi_{\leq t \leq}$ be the subposet of $(\Pi_n)^{\Sigma_d}$ consisting of partitions that are comparable with t. Let $\Pi_{<t<} = \Pi_{\leq t \leq} \setminus \{t\}$. Then clearly we have decompositions

$$|\Pi_{\leq T \leq}^{\Sigma_d}| = \bigcup_{t \in T} |\Pi_{\leq t \leq}^{\Sigma_d}|$$

and

$$(|\Pi_n^{\Sigma_d} \setminus T) \cap \Pi_{\leq T \leq}^{\Sigma_d}| \cong |\Pi_{<T<}^{\Sigma_d}| = \bigcup_{t \in T} |\Pi_{<t<}^{\Sigma_d}|.$$

It is clear that if t_1, t_2 are distinct elements of T then

$$\Pi_{\leq t_1 \leq}^{\Sigma_d} \cap \Pi_{\leq t_2 \leq}^{\Sigma_d} \subset (\Pi_n)^{\Sigma_d} \setminus T.$$

It follows that there is a homeomorphism

$$|\Pi_{\leq T \leq}^{\Sigma_d}|/|\Pi_n^{\Sigma_d} \setminus T) \cap \Pi_{\leq T \leq}^{\Sigma_d}| \cong \bigvee_{t \in T} |\Pi_{\leq t \leq}^{\Sigma_d}|/|\Pi_{<t<}^{\Sigma_d}|.$$

There are decompositions

$$|\Pi_{\leq t \leq}^{\Sigma_d}| \cong \{t\} * |\Pi_{<t}|^{\Sigma_d} * |\Pi_{>t}|^{\Sigma_d}$$

and

$$|\Pi_{<t<}^{\Sigma_d}| \cong |\Pi_{<t}|^{\Sigma_d} * |\Pi_{>t}|^{\Sigma_d}.$$

It follows that

$$|\Pi_{\leq t \leq}^{\Sigma_d}|/|\Pi_{<t<}^{\Sigma_d}| \cong \Sigma |\Pi_{<t}|^{\Sigma_d} * |\Pi_{>t}|^{\Sigma_d}.$$

Therefore, we obtain a homeomorphism,

$$|\Pi_{\leq T \leq}^{\Sigma_d}|/|(\Pi_n)^{\Sigma_d} \setminus T) \cap \Pi_{\leq T \leq}^{\Sigma_d}| \cong \bigvee_{t \in T} \Sigma |\Pi_{t<}|^{\Sigma_d} * |\Pi_{>t}|^{\Sigma_d}.$$

Recall that each $t \in T$ is a Σ_d-invariant partition of $\mathbf{n}$ with the property that each class of the partition intersects each Σ_d-orbit at exactly one

element. In particular, t has d equivalence classes, that are permuted transitively by Σ_d, with each class containing $\frac{n}{d}$ elements. It follows that $\Pi_{<t}^{\Sigma_d}$, the poset of Σ_d-invariant coarsening of t, is isomorphic to $\Pi_d^{\Sigma_d}$, which is in fact empty. On the other hand $\Pi_{t<}^{\Sigma_d}$, the poset of G-invariant refinements of t, is, by Proposition 2.8, isomorphic to $(\Sigma^{d-2} * \Pi_{\frac{n}{d}}^{*d})^{\Sigma_d}$, where Σ_d acts on Σ^{d-2} with no fixed points, and acts on $\Pi_{\frac{n}{d}}^{*d}$ by permuting the d copies of $\Pi_{\frac{n}{d}}$. The fixed points space of this action is $\Pi_{\frac{n}{d}}$. It follows that

$$|\Pi_n|^{\Sigma_d} \simeq T_+ \wedge \Sigma |\Pi_d|^{\Sigma_d} * |\Pi_{\frac{n}{d}}|.$$

In view of isomorphism (7.15), we proved that in the case $G = \Sigma_d$, the space $|\Pi_n|^G$, which is the target of the map (7.11) is homotopy equivalent to the source of that same map. The source of the map also can be decomposed as a pushout analogous to (7.16). The map respects the decomposition and induces a termwise equivalence of the pushout diagrams, therefore it induces an equivalence between homotopy pushouts. $\qquad\square$

8. Proof of the main theorem

As usual, let $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \subset \Sigma_n$ be a Young subgroup. We need to classify the isotypical isotropy groups of Π_n, considered as a space with an action of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \subset \Sigma_n$.

Lemma 8.1. *Consider the action of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$ on Π_n. Let $G \subset \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$ be an isotropy group of a simplex in Π_n. Suppose that G acts isotypically on $\mathbf{n}$. Then G is conjugate to a group of the form $\Sigma_{d_l} \wr \cdots \wr \Sigma_{d_1}$, where $l \geq 1$, and $d_1 \cdots d_l | \gcd(n_1, \ldots, n_k)$. As usual, the group G acts diagonally on blocks of size $d_1 \cdots d_l$. The possibility that G is the trivial group is included.*

Proof. Let $\lambda_1 < \cdots < \lambda_i$ be a chain of partitions of $\mathbf{n}$. Let G be the stabilizer of this chain in $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \subset \Sigma_n$. Let us analyze the general form of G. The group G acts on the set of equivalence classes of each λ_j. Recall that λ_1 is the coarsest partition of the chain. Let us say that two equivalence classes of λ_1 are of the same type, if they are in the same orbit of the action of G. This is an equivalence relation on the set of equivalence classes of λ_1. Let us say that there are s different types of equivalence classes of λ_1, and there are t_1 classes of type 1, t_2 classes of type 2 and so forth. It is easy to see that in this case G is isomorphic to a group of the following form

$$G \cong K_1 \wr \Sigma_{t_1} \times \cdots \times K_s \wr \Sigma_{t_s}$$

where K_j is the group of automorphisms (in $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$) of the restriction of the chain $\lambda_1 < \cdots < \lambda_i$ to an equivalence class of λ_1 of type j. We conclude that if G acts isotypically on $\mathbf{n}$ then either G is trivial, or $s = 1$ and $G \cong K_1 \wr \Sigma_{t_1}$. Furthermore, if G acts isotypically on $\mathbf{n}$ then K_1 has to act isotypically on a component of λ_1. The lemma follows by an induction on i. □

Now we are ready to prove the main theorem.

Proof of Theorem 1.2. We need to show that the $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-equivariant map

$$(8.2) \quad \Pi_n \longrightarrow \bigvee_{d|g} \left(\bigvee_{B(\frac{n_1}{d},...,\frac{n_k}{d})} \Sigma_{n_1} \times \cdots \times \Sigma_{n_k+} \wedge_{\Sigma_d} S^{n-d-1} * \Pi_d \right)$$

that we constructed in Proposition 6.10 is a $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$-equivariant homotopy equivalence. This means that we need to prove that the map induces a homotopy equivalence of G-fixed points, for every subgroup $G \subset \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$. By Lemma 2.15, it is enough to prove it for groups G that occur as isotropy groups of either Π_n (with respect to the action of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$) or of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k+} \wedge_{\Sigma_d} S^{n-d-1} * \Pi_d$.

Suppose first that G is not an isotypical subgroup of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$. Then $|\Pi_n|^G$ is contractible by Lemma 1.6. On the other hand, the fixed point space $\left(\Sigma_{n_1} \times \cdots \times \Sigma_{n_k+} \wedge_{\Sigma_d} S^{n-d-1} * \Pi_d \right)^G$ is contractible if G is not conjugate to a subgroup of Σ_d. If G is a subgroup of Σ_d then $\left(\Sigma_{n_1} \times \cdots \times \Sigma_{n_k+} \wedge_{\Sigma_d} S^{n-d-1} * |\Pi_d| \right)^G$ is homeomorphic to a wedge sum of some copies of $\left(S^{n-d-1} * |\Pi_d| \right)^G$ (Corollary 2.19). If $G \subset \Sigma_d \subset \Sigma_n$ does not act isotypically on $\mathbf{n}$, then it does not act isotypically on $\mathbf{d}$, and therefore $\left(S^{n-d-1} * |\Pi_d| \right)^G \cong (S^{n-d-1})^G * |\Pi_d|^G$ is contractible. We conclude that if G is not isotypical, then both the source and the target of the map (8.2) are contractible, and in particular the map is a homotopy equivalence.

It remains to consider the case when G is an isotypical group that occurs as an isotropy group of the action of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$ on Π_n and on $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k+} \wedge_{\Sigma_d} S^{n-d-1} * |\Pi_d|$. By Lemma 8.1, isotypical isotropy groups of $|\Pi_n|$ are groups of the form $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$. On the other side of the map, the isotropy groups of the action of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}$ on $\Sigma_{n_1} \times \cdots \times \Sigma_{n_k+} \wedge_{\Sigma_d} S^{n-d-1} * |\Pi_d|$ are the same as the isotropy groups of the action of Σ_d on $S^{n-d-1} * |\Pi_d|$. The isotropy groups of S^{n-d-1} are groups of the form $\Sigma_{d_1} \times \cdots \times \Sigma_{d_j}$ (Young subgroups of Σ_d). Therefore the isotropy groups of the action of Σ_d on $S^{n-d-1} * |\Pi_d|$ are the isotropy groups of the action of Young subgroups of Σ_d on $|\Pi_d|$.

Again by Lemma 8.1, the only isotypical isotropy groups are groups of the form $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$. So we need to check that the map (8.2) induces an equivalence of fixed points of groups of the form $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$ (including the trivial group).

For trivial isotopy group, the map is a cohomology isomorphism by Proposition 6.10. Since both the source and the target of the map are equivalent to wedges of spheres of dimension $n-3$, it follows that the map is a homotopy equivalence except possibly when $n = 4$. But this case can easily be checked "by hand". Indeed, Figure 1.4 illustrates the case of the group $\Sigma_2 \times \Sigma_2$, which is the most interesting Young subgroup of Σ_4.

Assume therefore that the isotropy group is non-trivial.

If $l > 1$, then $|\Pi_n|^{\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}}$ is contractible by Lemma 7.8. On the other hand $\left(\Sigma_{n_1} \times \cdots \times \Sigma_{n_k +} \wedge_{\Sigma_d} S^{n-d-1} * |\Pi_d|\right)^{\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}}$ is contractible if $\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}$ is not (conjugate to) a subgroup of Σ_d, or is equivalent to a wedge sum of copies of $\left(S^{n-d-1} * \Pi_d\right)^{\Sigma_{d_1} \wr \cdots \wr \Sigma_{d_l}}$, which is, again, contractible by Lemma 7.8.

It remains to consider isotropy groups of the form Σ_{d_1}, where d_1 is some number greater than 1 that divides $n_1, \ldots, n_k$. Fix such a number. We need to show that the map (8.2) induces a homotopy equivalence of Σ_{d_1} fixed points spaces:

(8.3)
$$|\Pi_n|^{\Sigma_{d_1}} \longrightarrow \bigvee_{d|g} \bigvee_{B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})} \left(\Sigma_{n_1} \times \cdots \times \Sigma_{n_k +} \wedge_{\Sigma_d} S^{n-d-1} * |\Pi_d|\right)^{\Sigma_{d_1}}.$$

By Lemma 7.12, $|\Pi_n|^{\Sigma_{d_1}}$ is homotopy equivalent to

$$\left(C(\Sigma_{d_1})^{\frac{n}{d_1}}\right)_+ \wedge_{C(\Sigma_{d_1})} \Sigma|\Pi_{\frac{n}{d_1}}|.$$

Here $C(\Sigma_{d_1})$ is the center of Σ_{d_1}, which is trivial if $d_1 > 2$ and is Σ_2 if $d_1 = 2$.

On the other hand $\left(\Sigma_{n_1} \times \cdots \times \Sigma_{n_k +} \wedge_{\Sigma_d} S^{n-d-1} * |\Pi_d|\right)^{\Sigma_{d_1}}$ is contractible if d_1 does not divide d. If $d_1|d$, then by Corollary 2.19 the fixed points space is equivalent to

(8.4) $\qquad C_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}(\Sigma_{d_1})_+ \wedge_{C_{\Sigma_d}(\Sigma_{d_1})} \left(S^{n-d-1} * |\Pi_d|\right)^{\Sigma_{d_1}}.$

The centralizer $C_{\Sigma_d}(\Sigma_{d_1})$ is isomorphic to $\Sigma_{\frac{d}{d_1}} \ltimes C(\Sigma_{d_1})^{\frac{d}{d_1}}$. Similarly,

$$C_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}(\Sigma_{d_1}) \cong \Sigma_{\frac{n_1}{d_1}} \times \cdots \times \Sigma_{\frac{n_k}{d_1}} \ltimes C(\Sigma_{d_1})^{\frac{n}{d_1}}.$$

It follows that the fixed points space (8.4) is equivalent to

$$(8.5)\quad \Sigma_{\frac{n_1}{d_1}} \times \cdots \times \Sigma_{\frac{n_k}{d_1}} \ltimes C(\Sigma_{d_1})^{\frac{n}{d_1}}_+ \wedge_{\Sigma_{\frac{d}{d_1}} \ltimes C(\Sigma_{d_1})^{\frac{d}{d_1}}} (S^{n-d-1} * |\Pi_d|)^{\Sigma_{d_1}}.$$

Clearly $(S^{n-d-1})^{\Sigma_{d_1}} \cong S^{\frac{n}{d_1} - \frac{d}{d_1} - 1}$. By Lemma 7.12,

$$|\Pi_d|^{\Sigma_{d_1}} \cong C(\Sigma_{d_1})^{\frac{d}{d_1}}_+ \wedge_{C(\Sigma_{d_1})} \Sigma|\Pi_{\frac{d}{d_1}}|$$

It follows that (8.5) can be rewritten as

$$\Sigma_{\frac{n_1}{d_1}} \times \cdots \times \Sigma_{\frac{n_k}{d_1}} \ltimes C(\Sigma_{d_1})^{\frac{n}{d_1}}_+ \wedge_{\Sigma_{\frac{d}{d_1}} \times C(\Sigma_{d_1})} (S^{\frac{n}{d_1} - \frac{d}{d_1} - 1} * \Sigma|\Pi_{\frac{d}{d_1}}|).$$

This can be rewritten, as a space, in the following form

$$C(\Sigma_{d_1})^{\frac{n}{d_1}} / C(\Sigma_{d_1})_+ \wedge \Sigma_{\frac{n_1}{d_1}} \times \cdots \times \Sigma_{\frac{n_k}{d_1}+} \wedge_{\Sigma_{\frac{d}{d_1}}} (S^{\frac{n}{d_1} - \frac{d}{d_1} - 1} * \Sigma|\Pi_{\frac{d}{d_1}}|).$$

It follows that the map (8.3) is equivalent to a map of the following form, suspended, and (if $d_1 = 2$) smashed with $C(\Sigma_{d_1})^{\frac{n}{d_1}}_+$ over $C(\Sigma_{d_1})$

$$(8.6)\quad \Pi_{\frac{n}{d_1}} \to \bigvee_{d'|\frac{g}{d_1}} \bigvee_{B(\frac{n_1}{d'd_1}, \dots, \frac{n_k}{d'd_1})} \Sigma_{\frac{n_1}{d_1}} \times \cdots \times \Sigma_{\frac{n_k}{d_1}+} \wedge_{\Sigma_{d'}} S^{\frac{n}{d_1} - d' - 1} * |\Pi_{d'}|.$$

We showed in Proposition 6.10 that there is a homotopy equivalence between these spaces (more precisely, we showed that there is a cohomology equivalence, which is a homotopy equivalence because the spaces are simply-connected except in the case $\frac{n}{d_1} = 4$, which can be checked directly). A tedious but straightforward inspection of all the maps involved shows the map (8.3) is the same as the one that was proved to be an equivalence in Proposition 6.10. $\square$

9. APPLICATION: FIXED POINTS

Let $G \subset \Sigma_n$ be a subgroup. We know that if G is not isotypical then $|\Pi_n|^G$ is contractible. Suppose G is isotypical. Then we may assume that G is a transitive subgroup of $\Sigma_d \subset (\Sigma_d)^{\frac{n}{d}} \subset \Sigma_n$, for some d that divides n. The following proposition describes the fixed points of an action of an isotypical group in terms of the fixed points of an action of a transitive group (which was analyzed in Lemma 7.2)

Proposition 9.1. *The map that was constructed in* (7.11)

$$C_{\Sigma_d}(G) \times \cdots \times C_{\Sigma_d}(G) \wedge_{C_{\Sigma_d}(G)} \Sigma|\Pi_d|^G * |\Pi_{\frac{n}{d}}| \to |\Pi_n|^G.$$

is a homotopy equivalence for all transitive subgroups G of Σ_d.

Proof. By theorem 1.2 there is a $(\Sigma_d)^{\frac{n}{d}}$-equivariant equivalence

$$\Pi_n \longrightarrow \bigvee_{d'|d} \bigvee_{B(\frac{d}{d'},...,\frac{d}{d'})} \Sigma_d \times \cdots \times \Sigma_{d+} \wedge_{\Sigma_{d'}} S^{n-d'-1} * |\Pi_{d'}|.$$

Since $G \subset (\Sigma_d)^{\frac{n}{d}}$, this map induces an equivalence of G-fixed points. Since G is a transitive subgroup of Σ_d, the fixed point space

$$\left(\Sigma_d \times \cdots \times \Sigma_{d+} \wedge_{\Sigma_{d'}} S^{n-d'-1} * |\Pi_{d'}| \right)^G$$

is contractible for $d' < d$. On the other hand, for $d' = d$ the fixed points space is homeomorphic to

$$\bigvee_{B(1,...,1)} C_{\Sigma_d}(G)^{\frac{n}{d}}_+ \wedge_{C_{\Sigma_d}(G)} (S^{n-d-1} * \Pi_d)^G \cong \bigvee_{B(1,...,1)} C_{\Sigma_d}(G)^{\frac{n}{d}}_+ \wedge_{C_{\Sigma_d}(G)} S^{\frac{n}{d}-2} * |\Pi_d|^G.$$

So we need to show that the following composed map is a homotopy equivalence

$$C_{\Sigma_d}(G)^{\frac{n}{d}}_+ \wedge_{C_{\Sigma_d}(G)} \Sigma |\Pi_{\frac{n}{d}}| * |\Pi_d|^G \to \bigvee_{B(1,...,1)} C_{\Sigma_d}(G)^{\frac{n}{d}} \wedge_{C_{\Sigma_d}(G)} S^{\frac{n}{d}-2} * |\Pi_d|^G.$$

A bit of diagram chasing confirms that this map is induced by the map

$$|\Pi_{\frac{n}{d}}| \to \bigvee_{B(1,...,1)} S^{\frac{n}{d}-3}$$

that collapses to a point the complement of the maximal simplices of $|\Pi_{\frac{n}{d}}|$ that correspond to the elements of $B(1,\ldots,1)$. The collapse map is a homotopy equivalence, and therefore the map on the fixed points is a homotopy equivalence. $\square$

Remark 9.2. It is interesting to compare our results with those of [3]. In that paper, the authors analyzed the fixed point space of $|\Pi_n|^P$ where P is a p-subgroup of Σ_n. More precisely, the following result was proved: the only p-subgroups $P \subset \Sigma_n$ for which the fixed points space $|\Pi_n|^P$ is not contractible are elementary abelian subgroups acting freely on $\mathbf{n}$.

It also was observed in [loc. cit.] that if $n = p^k$ and $P \cong (\mathbb{Z}/p)^k$ is an elementary abelian group acting freely and transitively on $\mathbf{n}$, then Π_n^P is isomorphic to the poset of proper non-trivial subgroups of P, which is closely related to the Tits building for $\mathrm{GL}_k(\mathbb{F}_p)$. Let T_k be the geometric realization of the poset of proper non-trivial subgroups of P. It is well-known that

$$T_k \simeq \bigvee_{p^{\binom{k}{2}}} S^{k-2}.$$

Now we can complete the calculation of fixed points for all p-subgroups of Σ_n. The following is an immediate consequence of Proposition 9.1.

Corollary 9.3. *Suppose $P \cong (\mathbb{Z}/p)^k$ is an elementary abelian group acting freely on $\mathbf{n}$. Write $n = mp^k$ (note that p may divide m). The fixed points space of the action of P on $|\Pi_n|$ is given by any one of the following formulas*

$$|\Pi_n|^P \simeq (\mathbb{Z}/p)^{km}_+ \wedge_{(\mathbb{Z}/p)^k} \Sigma T_k * |\Pi_m| \simeq \bigvee_{p^{k(m-1)+\binom{k}{2}}(m-1)!} S^{m+k-3}.$$

10. APPLICATION: ORBIT SPACES

In the previous section we showed how Theorem 1.2 can be used to analyze the fixed points of the action of a subgroup of Σ_n on Π_n. In this section we will apply the theorem to obtain information about the orbit space of the action of a Young subgroup on Π_n. Indeed, Theorem 1.2 has the following immediate corollary.

Proposition 10.1. *Suppose $n = n_1 + \cdots + n_k$. Let $g = \gcd(n_1, \ldots, n_k)$. There is a homotopy equivalence*

$$_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} \backslash \Pi_n \longrightarrow \bigvee_{d|g} \bigvee_{B(\frac{n_1}{d}, \ldots, \frac{n_k}{d})} {}_{\Sigma_d} \backslash \left(S^{n-d-1} \wedge |\Pi_d|^\diamond \right).$$

Corollary 10.2. *If $\gcd(n_1, \ldots, n_k) = 1$ then*

$$_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} \backslash |\Pi_n| \simeq \bigvee_{B(n_1, \ldots, n_k)} S^{n-3}.$$

The proposition reduces the problem of analyzing the space $_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} \backslash |\Pi_n|$ to spaces of the form $_{\Sigma_d} \backslash \left(S^{ld-1} \wedge |\Pi_d|^\diamond \right)$, where $l \geq 1$. Lemma 10.5 below will tell us that we may as well assume $l \geq 2$ (in particular, we may assume that $_{\Sigma_d} \backslash \left(S^{ld-1} \wedge |\Pi_d|^\diamond \right)$ is simply connected). First we need an auxiliary lemma.

Lemma 10.3. *Let $Y := \Sigma_{d_1} \times \cdots \times \Sigma_{d_i}$ be a non-trivial Young subgroup of of Σ_d, where $d = d_1 + \cdots + d_i$. Let N be the normalizer of Y and let W be any group satisfying $Y \subseteq W \subseteq N$. Equip S^{d-1} with the standard action of Σ_d. Then the orbit space $_W \backslash S^{d-1}$ is contractible.*

Proof. To begin with, we claim that the orbits space $_{\Sigma_d} \backslash S^{d-1}$ is contractible for $d > 1$. This is well-known. One way to see this is to identify S^{d-1} with $|\overline{\mathcal{B}_d}|^\diamond$, where $\overline{\mathcal{B}_d}$ is the poset of proper, non-trivial subsets of $\mathbf{d}$ (Example 2.2). Then

$$_{\Sigma_d} \backslash S^{d-1} \cong {}_{\Sigma_d} \backslash |\overline{\mathcal{B}_d}|^\diamond.$$

It is easy to see that quotient of the simplicial nerve of $\mathcal{B}_d$ by the action of Σ_d is isomorphic to the nerve of the linear poset $1 < 2 < \cdots < d-1$. In particular, its geometric realization is contractible.

For the general case, observe that there is a Y-equivariant homeomorphism $S^{d-1} \cong S^{d_1-1} * \cdots * S^{d_i-1}$. It follows that

$$_Y\backslash S^{d-1} \cong {}_{\Sigma_{d_1}}\backslash S^{d_1-1} * \cdots * {}_{\Sigma_{d_i}}\backslash S^{d_i-1}.$$

By our assumption at least one of the d_js is greater than one, and so the right hand side is a contractible space. Moreover, we claim that it is contractible as a N/Y-equivariant space. In other words, we claim that for every subgroup $H \subset N/Y$, the fixed point space

(10.4) $$\left({}_{\Sigma_{d_1}}\backslash S^{d_1-1} * \cdots * {}_{\Sigma_{d_i}}\backslash S^{d_i-1} \right)^H$$

is contractible. To see this, observe that N/Y is a Young subgroup of Σ_i (recall that i is the number of factors Σ_{d_j} of Y). N/Y acts on the space

$$S^{d-1} \cong {}_{\Sigma_{d_1}}\backslash S^{d_1-1} * \cdots * {}_{\Sigma_{d_i}}\backslash S^{d_i-1}$$

by permuting join factors that happen to be homeomorphic. It follows that the fixed point space (10.4) is a join of factors of the form $_{\Sigma_{d_j}}\backslash S^{d_j-1}$. Once again, at least one of the d_j is greater than 1, so at least one of these factors is contractible, and therefore the whole space is contractible.

It follows that the orbit space

$$_H\backslash \left({}_Y\backslash S^{d-1} \right)$$

is contractible for every $H \subset N/Y$. Finally, it follows that for every $Y \subset W \subset N$, the orbit space $_W\backslash S^{d-1}$ is contractible. $\qquad\square$

Lemma 10.5. *The space $_{\Sigma_d}\backslash \left(S^{d-1} \wedge |\Pi_d|^\diamond \right)$ is contractible for $d > 1$.*

Proof. The space $_{\Sigma_d}\backslash \left(S^{d-1} \wedge |\Pi_d|^\diamond \right)$ is a pointed homotopy colimit of spaces of the form $_G\backslash S^{d-1}$, where G is an isotropy group of $|\Pi_d|^\diamond$. We claim that the space $_G\backslash S^{d-1}$ is contractible for every G that occurs. From the claim it follows that the pointed homotopy colimit is contractible.

It remains to prove the claim. Let G be an isotropy group of $|\Pi_d|^\diamond$. Then either $G = \Sigma_d$ or G is the stabilizer group of a chain of proper non-trivial partitions of $\mathbf{d}$. In the first case,

$$_G\backslash S^{d-1} = {}_{\Sigma_d}\backslash S^{d-1} \simeq *.$$

In the second case, suppose that G is the stablizer of the chain of partitions $\lambda_0 < \cdots < \lambda_l$. By our convention, λ_l is the finest partition in the chains. Suppose λ_l has i equivalence classes, of sizes $d_1, \ldots, d_i$. Let

$Y \cong \Sigma_{d_1} \times \cdots \times \Sigma_{d_i}$ be the group that leaves invariant the equivalence classes of λ_l. Then G contains Y and is contained in the normalizer of Y, and $_G\backslash S^{d-1}$ is contractible by Lemma 10.3. $\qquad \square$

The following result concerns rational homology of these spaces.

Lemma 10.6. *The space* $_{\Sigma_d}\backslash\big(S^{ld-1} \wedge |\Pi_d|^\diamond\big)$ *has the rational homology of a point, unless* $d = 1$ *or* $d = 2$ *and* l *is even.*

Proof. Let us form the pointed homotopy orbit space

$$_{h\Sigma_d}\big(S^{ld-1} \wedge |\Pi_d|^\diamond\big) := E\Sigma_{d+} \wedge_{\Sigma_d} \big(S^{ld-1} \wedge |\Pi_d|^\diamond\big).$$

There is a natural map

$$_{h\Sigma_d}\big(S^{ld-1} \wedge |\Pi_d|^\diamond\big) \longrightarrow {}_{\Sigma_d}\backslash\big(S^{ld-1} \wedge |\Pi_d|^\diamond\big)$$

which is well-known to induce an isomorphism on rational homology. Therefore it is enough to prove the lemma with the homotopy orbit space replacing the strict orbit space. This was done in [5, 1]. For the sake of completeness we will sketch a proof.

The space $_{h\Sigma_d}\big(S^{ld-1} \wedge |\Pi_d|^\diamond\big)$ is a pointed homotopy colimit of spaces of the form $_{hG}S^{ld-1}$, where G is an isotropy group of $|\Pi_d|^\diamond$. By an easy Serre spectral sequence argument, such a space is rationally trivial if l is odd and G contains a transposition. It is easy to see that every isotropy group of $|\Pi_d|^\diamond$ contains a transposition, so we are done in the case when l is odd.

If l is even then the homology of $_{h\Sigma_d}\big(S^{ld-1} \wedge |\Pi_d|^\diamond\big)$ is isomorphic, with dimension shift of $ld - 1$, to the homology of $_{h\Sigma_d}|\Pi_d|^\diamond$, by Thom isomorphism. The space $_{h\Sigma_d}|\Pi_d|^\diamond$ is rationally equivalent to $_{\Sigma_d}\backslash|\Pi_d|^\diamond$, and the latter space is contractible by Kozlov's theorem [16].

$\qquad \square$

Remark 10.7. Let us consider the case $d = 2$. In this case $|\Pi_2| = \emptyset$, so

$$_{\Sigma_2}\big(S^{2l-1} \wedge \Pi_2|^\diamond\big) \cong {}_{\Sigma_2}S^{2l-1} \cong \Sigma^l\mathbb{R}P^{l-1}.$$

This space is rationally contractible if l is odd. Note that if l is even, the reduced homology of $\Sigma^l\mathbb{R}P^{l-1}$ has a copy of $\mathbb{Z}$ in dimension $2l - 1$, and is 2-torsion otherwise.

Corollary 10.8. *The torsion free part of* $\widetilde{\mathrm{H}}_*(_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}\backslash\Pi_n)$ *is concentrated in degree* $n - 3$. *If* $2|\gcd(n_1, \ldots, n_k)$ *and* $\frac{n}{2}$ *is odd, then the torsion-free part is isomorphic to*

$$\mathbb{Z}^{B(n_1,\ldots,n_k)} \oplus \mathbb{Z}^{B(\frac{n_1}{2},\ldots,\frac{n_k}{2})}.$$

In all other cases, the torsion-free part is $\mathbb{Z}^{B(n_1,\ldots,n_k)}$.

Proof. This follows easily from Proposition 10.1 and Lemma 10.6. $\quad\square$

We also have the following companion to the rational calculation.

Lemma 10.9. *The groups* $\widetilde{\mathrm{H}}_*({}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} \backslash \Pi_n)$ *have p-primary torsion only for primes that satisfy* $p \leq \gcd(n_1, \ldots, n_k)$.

Proof. The space ${}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} \backslash \Pi_n$ is a wedge sum of spaces of the form ${}_{\Sigma_d} \backslash (S^{ld-1} \wedge |\Pi_d|^\diamond)$, where d divides $\gcd(n_1, \ldots, n_k)$. So it is enough to show that this space has no p-primary torsion for $p > d$. The cases $d = 1, 2$ can easily be checked "by hand". Let us suppose that $d > 2$ and $p > d$. To prove that the homology of ${}_{\Sigma_d} \backslash (S^{ld-1} \wedge |\Pi_d|^\diamond)$ has no p-primary torsion it is enough to prove that its reduced homology groups with $\mathbb{Z}/p$ coefficients are zero. Since $p > d$, the map

$$
{}_{h\Sigma_d} \backslash (S^{ld-1} \wedge |\Pi_d|^\diamond) \to {}_{\Sigma_d} \backslash (S^{ld-1} \wedge |\Pi_d|^\diamond)
$$

induces an isomorphism on homology with $\mathbb{Z}/p$ coefficients. So it is enough to prove that the homotopy orbits space ${}_{h\Sigma_d} \backslash (S^{ld-1} \wedge |\Pi_d|^\diamond)$ has no reduced homology with $\mathbb{Z}/p$ coefficients. In fact, it follows from [2] that this space is contractible unless d is a prime power, and is p-local if $d = p^k$. But for the reader's convenience we will give a relatively elementary self-contained proof of the result that we need. We saw in the proof of Lemma 10.6 that the reduced homology of this space is all torsion, so it is enough to show that it has no p-primary torsion. To see this consider the following homomorphisms
(10.10)
$$
\widetilde{\mathrm{H}}_* \left({}_{h\Sigma_d} \backslash \left(S^{ld-1} \wedge |\Pi_d|^\diamond \right) \right) \to \widetilde{\mathrm{H}}_* \left({}_{h\Sigma_{d-1}} \backslash \left(S^{ld-1} \wedge |\Pi_d|^\diamond \right) \right) \to \widetilde{\mathrm{H}}_* \left({}_{h\Sigma_d} \backslash \left(S^{ld-1} \wedge |\Pi_d|^\diamond \right) \right).
$$

Here the first homomorphism is the transfer in homology associated with the inclusion $\Sigma_{d-1} \hookrightarrow \Sigma_d$, and the second homomorphism is induced by the quotient map. It is well-known that the composed homomorphism is multiplication by $d = |\Sigma_d / \Sigma_{d-1}|$. By the main theorem, $|\Pi_d|$ is Σ_{d-1}-equivariantly equivalent to $\Sigma_{d-1_+} \wedge S^{d-3}$. Therefore

$$
{}_{h\Sigma_{d-1}} \backslash \left(S^{ld-1} \wedge |\Pi_d|^\diamond \right) \simeq S^{ld+d-3}.
$$

The reduced homology of this space is $\mathbb{Z}$ in dimension $ld + d - 3$ and zero in all other dimensions. It follows that the composed homomorphism (10.10) is zero on torsion elements. So multiplication by d is zero on the torsion elements of the group $\widetilde{\mathrm{H}}_* \left({}_{h\Sigma_d} \backslash \left(S^{ld-1} \wedge |\Pi_d|^\diamond \right) \right)$. It follows that it is a d-torsion group and in particular it has no p-primary torsion for $p > d$. $\quad\square$

Our next task is to analyze the space $_{\Sigma_p}\backslash\left(S^{lp-1} * |\Pi_p|\right)$ in the case when p is a prime. In the case $p = 2$ we already saw that

$$_{\Sigma_2}\backslash\left(S^{2l-1} * |\Pi_2|\right) \cong \Sigma^l \mathbb{R}P^{l-1}.$$

Recall that there is a Σ_p-equivariant homeomorphism $S^{lp-1} \cong S^{l-1} \wedge (S^{p-1})^l$, where Σ_p acts on S^{p-1} via the reduced standard representation, and acts trivially on S^{l-1}. The fixed-points space $S^{lp-1\Sigma_p}$ is homeomorphic to S^{l-1}. Let $S^{l-1} \hookrightarrow S^{lp-1}$ be the inclusion of fixed points. The key to analyzing $_{\Sigma_p}\backslash\left(S^{lp-1} \wedge |\Pi_p|^\diamond\right)$ for a general prime p is the following proposition.

Proposition 10.11. *The following is a homotopy pushout square*

(10.12)
$$\begin{array}{ccc} _{h\Sigma_p}\backslash\left(S^{l-1} \wedge |\Pi_p|^\diamond\right) & \rightarrow & _{\Sigma_p}\backslash\left(S^{l-1} \wedge |\Pi_p|^\diamond\right) \\ \downarrow & & \downarrow \\ _{h\Sigma_p}\backslash\left(S^{lp-1} \wedge |\Pi_p|^\diamond\right) & \rightarrow & _{\Sigma_p}\backslash\left(S^{lp-1} \wedge |\Pi_p|^\diamond\right) \end{array}.$$

Proof. The homotopy cofiber of the map $S^{l-1} \rightarrow S^{lp-1}$ is equivalent to $S^l \wedge (S^{l(p-1)-1})_+$. Therefore, taking homotopy cofibers of the vertical maps we obtain the map

$$_{h\Sigma_p}\backslash\left(S^l \wedge (S^{l(p-1)-1})_+ \wedge |\Pi_p|^\diamond\right) \rightarrow _{\Sigma_p}\backslash\left(S^l \wedge (S^{l(p-1)-1})_+ \wedge |\Pi_p|^\diamond\right).$$

We want to prove that this map is a homotopy equivalence. For this, it is enough to prove that the following map is an equivalence

$$_{h\Sigma_p}\backslash(S^{l(p-1)-1})_+ \wedge |\Pi_p|^\diamond \rightarrow _{\Sigma_p}\backslash(S^{l(p-1)-1})_+ \wedge |\Pi_p|^\diamond.$$

The Σ_p-space $S^{l(p-1)-1}$ can be written as a homotopy colimit of sets of the form Σ_p/G, where G is an isotropy group of $S^{l(p-1)-1}$. Therefore, it is enough to prove that for every group G that occurs as an isotropy of $S^{l(p-1)-1}$, the map $_{h\Sigma_p}\backslash(\Sigma_p/G)_+ \wedge |\Pi_p|^\diamond \rightarrow _{\Sigma_p}\backslash(\Sigma_p/G)_+ \wedge |\Pi_p|^\diamond$ is an equivalence. We claim that every isotropy group of $S^{l(p-1)-1}$ is is contained in a group of the form $\Sigma_{p_1} \times \cdots \times \Sigma_{p_l}$, where $l > 1$ and $p_1 + \cdots + p_l = p$. Indeed, for $l = 1$, S^{p-2} is the boundary of the $p-1$-simplex, and it is easy to see that the isotropy groups have the form $\Sigma_{p_1} \times \cdots \times \Sigma_{p_l}$, where $l > 1$ and $p_1 + \cdots + p_l = p$. For $l > 1$ one may argue by induction on l. Since p is a prime, $\gcd(p_1, \ldots, p_l) = 1$. It follows, using our main theorem, that the action of $\Sigma_{p_1} \times \cdots \times \Sigma_{p_l}$ on Π_p is essentially pointed-free, in the sense that for every subgroup $H \subset \Sigma_{p_1} \times \cdots \times \Sigma_{p_l}$, the fixed points space Π_p^H is contractible. It follows that the action of G on Π_p is essentially pointed-free. It follows that the map $_{h\Sigma_p}\backslash(\Sigma_p/G)_+ \wedge |\Pi_p|^\diamond \rightarrow _{\Sigma_p}\backslash(\Sigma_p/G)_+ \wedge |\Pi_p|^\diamond$, which is the same as the map $_{hG}\backslash|\Pi_p|^\diamond \rightarrow _G\backslash|\Pi_p|^\diamond$, is an equivalence. $\square$

Corollary 10.13. *If p is an odd prime then there is a homotopy cofibration sequence*

$$_{h\Sigma_p}\backslash\left(S^{l-1}\wedge|\Pi_p|^\diamond\right) \to {}_{h\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|^\diamond\right) \to {}_{\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|^\diamond\right).$$

Proof. Consider the upper right corner of (10.12). Since $p > 2$, $_{\Sigma_p}\backslash\left(S^{l-1}\wedge|\Pi_p|^\diamond\right)$, which is homeomorphic to $S^{l-1}\wedge\left(_{\Sigma_p}\backslash|\Pi_p|^\diamond\right)$, is contractible by Kozlov's theorem [16]. The corollary now follows from Proposition 10.11. $\square$

Corollary 10.14. *If p is an odd prime then the space $_{\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|^\diamond\right)$ is p-local.*

Proof. By Corollary 10.13 it is enough to prove that the spaces $_{h\Sigma_p}\backslash\left(S^{l-1}\wedge|\Pi_p|^\diamond\right)$ and $_{h\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|^\diamond\right)$ are each p-local. Since $l \geq 1$ and $p > 2$ both of these spaces are simply connected. Therefore, it is enough to show that the integral homology of these spaces with $\mathbb{Z}$ coefficients is p-torsion. This can be done using a transfer argument similar to the one in the proof of Lemma 10.9. $\square$

Proposition 10.15. *Let p be an odd prime. If l is odd, then the orbits space $_{\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|^\diamond\right)$ is equivalent to the p-localization of $_{\Sigma_p}\backslash S^{lp-1}$. If l is even then $_{\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|^\diamond\right)$ is equivalent to the p-localization of the homotopy cofiber of the quotient map $S^{lp-1} \to {}_{\Sigma_p}\backslash S^{lp-1}$.*

Proof. Once again we use the homotopy cofibration sequence

$$_{\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|_+\right) \to {}_{\Sigma_p}\backslash S^{lp-1} \to {}_{\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|^\diamond\right).$$

Our task reduces to showing that if l is odd then the p-localization of the space $_{\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|_+\right)$ is trivial, and if l is even then for any choice of point in $|\Pi_p|$, the induced map $S^{lp-1} \to {}_{\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|_+\right)$ is a p-local equivalence.

Consider the pointed Σ_p-space $S^{lp-1}\wedge|\Pi_p|_+$. It is easy to check that every isotropy group of the space (aside from basepoint) is a non-transitive subgroup of Σ_p. It follows that p does not divide the order of any of the isotropy groups. It follows that for every isotropy group G, the map $BG \to *$ induces an isomorphism in homology with $\mathbb{Z}/p$-coefficients. It follows that the natural map $_{h\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|_+\right) \to {}_{\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|_+\right)$ is a mod p-homology isomorphism and therefore a p-local isomorphism. The space $_{h\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|_+\right)$ is a pointed homotopy colimit of spaces of the form $_{hG}\backslash S^{lp-1}$, where G is an isotropy group of $|\Pi_p|$. The relevant properties of G are that (1) p does not divide the order of G and (2) G contains a transposition. Suppose that l is odd. Then $\widetilde{H}_*\left(_{hG}\backslash S^{lp-1}; \mathbb{Z}/p\right) \cong \{0\}$, and so $_{hG}\backslash S^{lp-1}$ is p-locally trivial. It follows that $_{h\Sigma_p}\backslash\left(S^{lp-1}\wedge|\Pi_p|_+\right)$ is p-locally trivial. Now

suppose l is even. The space $_{h\Sigma_p}\backslash\left(S^{lp-1} \wedge |\Pi_p|_+\right)$ is a Thom space of a bundle over $E\Sigma_p \times_{\Sigma_p} |\Pi_p|$. Since l is even, the bundle is orientable, and therefore Thom isomorphism holds. Any map $* \to E\Sigma_p \times_{\Sigma_p} |\Pi_p|$ is a mod p homology isomorphism. Therefore by Thom isomorphism it induces a mod p homology isomorphism of Thom spaces $S^{lp-1} \to$ $_{h\Sigma_p}\backslash\left(S^{lp-1} \wedge |\Pi_p|_+\right)$. $\square$

Corollary 10.16. *Let p be a prime. For $l > 2$ the space $_{\Sigma_p}\backslash\left(S^{lp-1} \wedge |\Pi_p|^\diamond\right)$ is not equivalent to a wedge of spheres. For $l = 2$ it is equivalent to S^{2l-1} if $p = 2$ and is contractible if p is odd.*

Remark 10.17. Let us recall that for $l = 1$ the space is contractible by Lemma 10.5.

Proof. Consider first the case $p = 2$. In this case we identified $_{\Sigma_p}\backslash\left(S^{lp-1} \wedge |\Pi_p|^\diamond\right)$ with $\Sigma^l \mathbb{R}P^{l-1}$. For $l = 2$ this space is equivalent to S^3. For $l > 2$ it is not equivalent to a wedge of spheres since its homology has 2-torsion.

Suppose now that $p > 2$. By Proposition 10.15 we need to analyze the space $_{\Sigma_p}\backslash S^{lp-1}$ or the homotopy cofiber of the quotient map $S^{lp-1} \to$ $_{\Sigma_p}\backslash S^{lp-1}$. Note that the space $_{\Sigma_p}\backslash S^{lp-1}$ is a desuspension of $(S^l)^{\wedge p}_{\Sigma_p}$. The homology with $\mathbb{Z}/p$ coefficients of these spaces is known by work of Nakaoka [19]. Applying Nakaoka's result to our situation, we find that the The homology with $\mathbb{Z}/p$ coefficients of $_{\Sigma_p}\backslash\left(S^{lp-1} \wedge |\Pi_p|^\diamond\right)$ is generated by symbols $\{i\}$, where i is an integer satisfying the following constraints

- $i > 1$
- $i \equiv 0$ or $1 \,(\mathrm{mod}\, 2(p-1))$
- $i < (p-1)l$

The degree of $\{i\}$ is $l + i - 1$. It is easy to see that for $l \leq 2$ there are no integers i satisfying these constraints. It follows that for $l \leq 2$ the mod p homology of $_{\Sigma_p}\backslash\left(S^{lp-1} \wedge |\Pi_p|^\diamond\right)$ is trivial. Since the space is p-local, it follows that it is contractible.

On the other hand, for $l > 2$ it is easy to see that there is (more than one) value of i satisfying the constraints. It follows that the space is not contractible. Since it is p-local, it is not equivalent to a wedge of spheres. $\square$

Corollary 10.18. *The quotient space $_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}\backslash\Pi_n$ is homotopy equivalent to a wedge of spheres if and only if one of the following holds:*

- $\gcd(n_1, \ldots, n_k) = 1$.
- *p is a prime, $n = 2p$ or $3p$, and $\gcd(n_1, \ldots, n_k) = p$.*

Proof. The case $\gcd(n_1, \ldots, n_k) = 1$ was dealt with in Corollary 10.2. In the second case, Propositioin 10.1 tells us that $_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} \backslash \Pi_n$ is equivalent to a wedge sum of spaces of the form S^{n-3} and $_{\Sigma_p} \backslash (S^{n-p-1} \wedge |\Pi_p|^\diamond)$. Since $n = 2p$ or $3p$, $n - p = lp$ where $l = 1, 2$. By Lemma 10.5 and Corollary 10.16 all these spaces are either equivalent to a sphere or are contractible.

In all other cases, let p be the smallest prime that divides $\gcd(n_1, \ldots, n_k)$. Then p divides n and $\frac{n}{p} > 3$. It follows that $_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} \backslash \Pi_n$ has a wedge summand equivalent to

$$\bigvee_{B(\frac{n_1}{p}, \ldots, \frac{n_k}{p})} {}_{\Sigma_p} \backslash (S^{lp-1} \wedge |\Pi_p|^\diamond)$$

where $l > 2$. By Corollary 10.16 this space is not equivalent to a wedge of spheres. $\square$

Example 10.19. To illustrate our results, let us analyze the homotopy type and the homology groups of $_{\Sigma_4 \times \Sigma_4} \backslash |\Pi_8|^\diamond$. By Proposition 10.1 there is a homotopy equivalence

$${}_{\Sigma_4 \times \Sigma_4} \backslash |\Pi_8|^\diamond \simeq \bigvee_{B(4,4)} S^5 \vee \bigvee_{B(2,2)} {}_{\Sigma_2} \backslash S^5 \vee \bigvee_{B(1,1)} {}_{\Sigma_4} \backslash (S^3 \wedge |\Pi_4|^\diamond).$$

The last factor is contractible by Lemma 10.5. A quick calculation shows that $|B(4,4)| = 8$ and $|B(2,2)| = 1$. We already observed that $_{\Sigma_2} \backslash S^5 \cong \Sigma^3 \mathbb{R}P^2$. We conclude that there is a homotopy equivalence

$${}_{\Sigma_4 \times \Sigma_4} \backslash |\Pi_8|^\diamond \simeq \Sigma^3 \mathbb{R}P^2 \vee \bigvee_8 S^5.$$

It follows that the reduced homology of $_{\Sigma_4 \times \Sigma_4} \backslash |\Pi_8|^\diamond$ is isomorphic to $\mathbb{Z}/2$ in dimension 4, to $\mathbb{Z}^8$ in dimension 5, and is zero otherwise. This is consistent with computer calculations done by Donau [9].

REFERENCES

[1] G. Arone, *A note on the homology of Σ_n, the Schwartz genus, and solving polynomial equations*, An alpine anthology of homotopy theory, 1–10, Contemp. Math., **399**, Amer. Math. Soc., Providence, RI, 2006.
[2] G. Arone and W. Dwyer, *Partition complexes, Tits buildings and symmetric products*, Proc. London Math. Soc. (3) **82** (2001), no. 1, 229–256.
[3] G. Arone, W. Dwyer, and K. Lesh *Bredon homology of partition complexes*, arXiv:1306.0056.
[4] G. Arone and M. Kankaanrinta, *The homology of certain subgroups of the symmetric group with coefficients in $\mathcal{L}ie(n)$*, J. Pure Appl. Algebra **127** (1998), no. 1, 1–14.

[5]   G. Arone and M. Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Invent. Math. **135** (1999), no. 3, 743–788.

[6]   H. Barcelo, *On the action of the symmetric group on the free Lie algebra and the partition lattice*, J. Combin. Theory Ser. A 55 (1990), no. 1, 93–129.

[7]   L. Brantner, *Fixed points of the partition complex*, in preparation.

[8]   R. Donau, *A nice acyclic matching on the nerve of the partition lattice*, arXiv:1204.2693.

[9]   R. Donau, Personal communication.

[10] W. G. Dwyer, *Homology decompositions for classifying spaces of finite groups*, Topology **36** (1997), no. 4, 783–804.

[11] R. Freij, *Equivariant discrete Morse theory*, Discrete Math. **309** (2009), no. 12, 3821–3829.

[12] M. Hall, *A basis for free Lie rings and higher commutators in free groups*, Proc. Amer. Math. Soc. **1**, (1950) 575–581.

[13] P. Hall, *A Contribution to the Theory of Groups of Prime-Power Order*, Proc. London Math. Soc. **36** (1934) 29–95.

[14] P. Hersh, *Lexicographic shellability for balanced complexes*, J. Algebraic Combin. **17** (2003), no. 3, 225?254.

[15] M. J. Klass, *Enumeration of partition classes induced by permutation groups*, Ph.D. Dissertation, Department of Mathematics, UCLA, 1972.

[16] D. Kozlov, *Collapsibility of $\Delta(\Pi_n)/S_n$ and some related CW complexes*, Proc. Amer. Math. Soc. **128** (2000), no. 8, 2253–2259.

[17] M. Markl, S. Shnider, and J. Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, 96. American Mathematical Society, Providence, RI, 2002. x+349 pp.

[18] J. Milnor, *On the Construction FK*, 1956 Princeton University lecture notes. In J. F. Adams, Algebraic Topology, A Student's Guide, London Mathematical Society Lecture Note Series 4, Cambridge University Press, 1972.

[19] M. Nakaoka *Cohomology mod p of the p-fold symmetric products of spheres*, J. Math. Soc. Japan **9** (1957), 417–427.

[20] J. Neisendorfer, *Algebraic methods in unstable homotopy theory,* New Mathematical Monographs, 12. Cambridge University Press, Cambridge, 2010. xx+554 pp.

[21] D. Quillen, *Homotopy properties of the poset of nontrivial p-subgroups of a group*, Adv. in Math. **28** (1978), 101–128.

[22] C. Reutenauer, *Free Lie algebras*, London Mathematical Society Monographs. New Series, 7. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. xviii+269 pp.

[23] M. Wachs, *On the (co)homology of the partition lattice and the free Lie algebra*, Selected papers in honor of Adriano Garsia (Taormina, 1994). Discrete Math. 193 (1998), no. 1-3, 287–319.

[24] J. Walker, *Canonical homeomorphisms of posets*, European J. Combin. **9** (1988), no. 2, 97–107.

[25] D. White and S. G Williamson, *Combinatorial structures and group invariant partitions*, Proc. Amer. Math. Soc. **55** (1976), no. 1, 233–236.

[26] E. Witt, *Treue Darstellung Liescher Ringe*, J. Reine. Angew. Math **77** (1937), 152–160.

Kerchof Hall, U. of Virginia, P.O. Box 400137, Charlottesville VA 22904 USA

Current address: Department of Mathematics, Stockholm University, SE - 106 91 Stockholm, Sweden

E-mail address: `zga2m@virginia.edu`

URL: `http://www.math.virginia.edu/~zga2m`